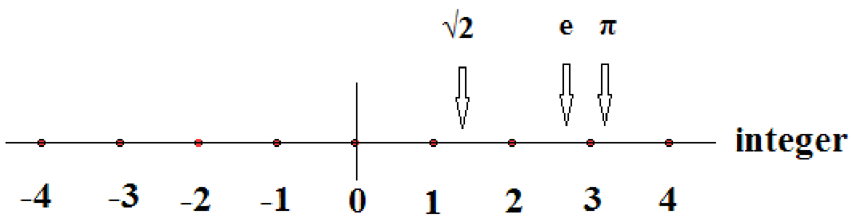
INTRODUCTION TO COMPLEX NUMBERS

The numbers … -4, -3, -2, -1, 0, 1, 2, 3, 4… represent the negative and positive real numbers termed integers. As one first learns in middle school they can be thought of as unit distance spaced points along a straight line as shown-



The basic mathematical operations with these numbers are:

addition : A + B subtraction: A-B multiplication: AxB and division: A/B

There are numbers which lie between these whole numbers. They are the rationals such as 52/73=0.7123287671232876712328767…whose digital expansion repeats itself after n units and the irrationals such as sqrt(2)= 1.414213562373024…where the digits never repeat themselves.

Although the product of two integers will always be an integer, the same is not true for the roots of integers. Take, for example, the square root of 2 which is a non-terminating irrational number. If one takes things one step further asks what is sqrt(-2), one finds-

$$\sqrt{-2} = (\sqrt{2})(\sqrt{-1}) \quad where \quad sqrt(2)\ is\ as\ given\ above$$

but what is the meaning of sqrt(-1)? One calls this quantity the imaginary number i . With this definition it is possible to extend the set of all numbers into an even larger set of <u>complex numbers</u> -

$$z=a+ib$$

with 'a' representing the real part and 'b' the imaginary part of the number z. The standard designation is a=Re(z) and b=Im(z). We also have that the absolute value of z equals Ab(z)=|z|=sqrt(a^2+b^2). On taking the square of z we have-

$$z^2=a^2+2iab+(ib)(ib)=(a^2-b^2)+2iab$$

since i^2=-1. Taking more powers of i we find the following properties-

$$i^0 = 1 \quad i^1 = i \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1$$

So that we can write any integer power p of i as-

$$i^p = i^{(4n+r)} = i^r \text{ with n integer and the remainder r=0, 1, 2 or 3}$$

That is $i^{359} = i^{(356+3)} = i^3 = -i$

The basic addition, subtraction, multiplication and division laws for complex numbers remain as they were for real numbers. Therefore-

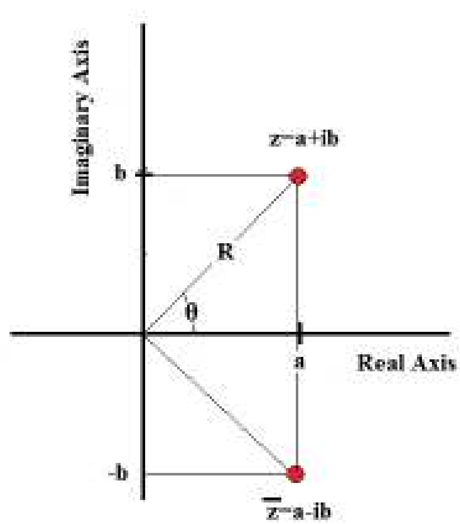
$$(1+i)^3 = 1+3i+3i^2+i^3 = 2(-1+i) \text{ and } (3-i)+(-2+2i) = 1+i$$

A convenient way to plot a complex number z is by means of an Argand Diagram in which the real part of a complex number is measured along the x axis and the imaginary portion measured along the y axis. We can represent z in either its Cartesian form or its polar form. They read respectively-

$$z = a+ib = \mathrm{sqrt}(a^2+b^2)\exp[i \arctan(b/a)] = R \exp(i\theta)$$

Here $|z| = R = \mathrm{sqrt}(a^2+b^{2)}$ **is the amplitude(or modulus) and** $\theta = \arg(z) = \arctan(b/a)$ **the argument of z. By replacing i by –i in a complex number one produces its complex conjugate designated by** $\overline{z}$**. One always has that** $z\,\overline{z} = |z|^2$ **is a real number.**
Here is a graph of the complex number z=a+ib and its conjugate in the Argand diagram-

ARGAND DIAGRAM FOR Z AND ZBAR



Let us next look at the number exp(z)=exp(a+ib) in more detail. Expanding this function as a Taylor series we have-

$$e^{a+ib} = e^a e^{ib} = e^a \left\{ (1 - \frac{b^2}{2!} + \frac{b^4}{4!} - ...) + i(b - \frac{b^3}{3!} + \frac{b^5}{5!} - ...) \right\}$$

But the two infinite series in the curly bracket are recognized as cos(b) and sin(b). Hence one has the famous formula first derived by Leonard Euler, namely,-

$$\exp(a + ib) = e^a [\cos(b) + i \sin(b)]$$

On setting a=0 and b=π/2 and π we have exp(iπ/2)=i and exp(iπ)=-1, respectively. Thus one can conclude that-

$$i^n = \exp(in\pi / 2) = \cos(n\pi / 2) + i \sin(n\pi / 2)$$

If n=i we have that-

$$i^i = \exp(-\pi / 2) = 0.20787957...$$

which is a real but irrational number. Also we have that z^3 +1=0 has the solution-

$$z = (-1)^{1/3} = i^{2/3} = \cos(\pi / 3) + i \sin(\pi / 3) = (1 + i\sqrt{3}) / 2$$

This, however, is not the only solution since there are two more which can be gotten by rotating away from the first solution by Δθ=±2π/3 radians. The other two solutions are-

cos(π)+isin(π)=-1 and cos(-π/3)+isin(-π/3))=(1-i√3)/2

The root of a complex number can be written as-

$$z^{1/N} = R^{1/N} \exp[i \frac{(\theta + 2\pi k)}{N}] \quad where \quad k = 0, \pm 1, \pm 2, etc$$

This equality is known in the literature as the deMoivre Formula. It follows from the form of the polar representation of the complex number z and clearly shows the presence of N multiple roots. A question I used to ask my undergraduate analysis class at the University of Florida was "Are there any real solutions to the pth root of i when p is an integer"? The answer is no and here is the proof-

$$i^{1/p}=\exp[i\pi\frac{(1+4k)}{2p}]$$

To have a real solution requires that sin[π(1+4k)/(2p)]=0 which implies that p=(1+4k)/2n with n=0,±1,±2 etc . We see that the numerator in this expression is an odd number and the denominator is always an even number. Hence p can never be an integer and so no pure real solutions can exist! Note that pure real roots are possible if p equals certain rational numbers such as 2/3.

One can derive numerous trigonometric identities using the Euler Formula as a starting point. First setting a =0 we have the results-

exp(ib)=cos(b)+isin(b) and exp(-ib)=cos(b)-isin(b)

Adding and subtracting these together, we arrive at the identities-

$$\cos(b)=\frac{(e^{ib}+e^{-ib})}{2}\quad and\quad \sin(b)=\frac{(e^{ib}-e^{-ib})}{2i}$$

which upon replacing b by ic produces the hyperbolic functions-

$$\cos(ic)=\frac{(e^{c}+e^{-c})}{2}=\cosh(c)\quad and\quad \sin(ic)=\frac{(e^{c}-e^{-c})}{2i}=-i\sinh(c)$$

Also, on letting b=A+B, we obtain the well known trigonometric identities-

$$\begin{aligned}\cos(A+B)&=\frac{1}{2}\{[\cos(A)+i\sin(A)][\cos(B)+i\sin(B)]+[\cos(A)-i\sin(A)][\cos(A)-i\sin(B)]\}\\&=\cos(A)\cos(B)-\sin(A)\sin(B)\end{aligned}$$

and-

$$\begin{aligned}\sin(A+B)&=\frac{1}{2i}\{[\cos(A)+i\sin(A)][\cos(B)+i\sin(B)]-[\cos(A)-i\sin(A)][\cos(B)-i\sin(B)]\}\\&=\sin(A)\cos(B)+\sin(B)\cos(A)\end{aligned}$$

The double angle formulas-

$$\cos(2A)=\cos^2 A-\sin^2 A=1-2\sin^2 A\quad and\quad \sin(2A)=2\sin(A)\cos(A)$$

follow on setting A=B. We can also use the complex number representations for sin(A) and cos(A) to develop the quadruple angle formulas-

$$\cos(4A) = 1 - 8\cos^2(A) + 8\cos^4(A) \quad and \quad \sin(4A) = 8\sin(A) - 24\sin^3(A) + 16\sin^5(A)$$

Certain definite integrals can also be nicely solved using complex numbers. Consider the integral-

$$K = \int_{x=0}^{\infty} \sin(ax)\exp(-bx)dx = \mathrm{Im}\int_{x=0}^{\infty} \exp(-b+ia)x\,dx \quad with \quad a \ and \ b > 0$$

Here Im stands for the imaginary part of the function just like Re would stand for the real part. After a simple integration of the exponential function we have-

$$K = \mathrm{Im}\left\{\frac{1}{b-ia}\right\} = \mathrm{Im}\left\{\frac{(b+ia)}{(b-ia)(b+ia)})\right\} = \frac{a}{b^2+a^2}$$

I remember how we derived this result many tears ago in my first college calculus class by a much longer route involving several integration by parts. A benefit of the complex number approach is not only its ease compared to other methods but also the fact that it will sometimes yield additional information such as, in this case, that-

$$L = \int_{x=0}^{\infty} \cos(ax)\exp(-bx) = \mathrm{Re}\int_{x=0}^{\infty} \exp(-b+ia)x\,dx = \frac{b}{b^2+a^2}$$

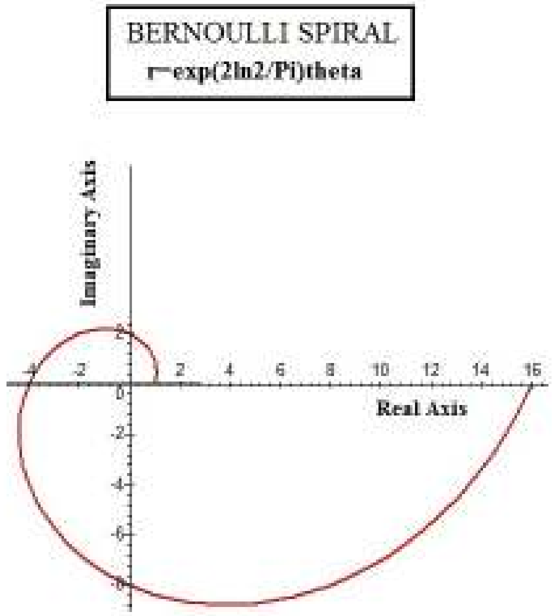
Finally let us show how one can plot a function such as $F=z^n=R^n\exp(in\theta)$ in the complex plane. Specifically let n be any positive power greater than one including non integer values. Also set z=a+ib . On substituting these value into F we find –

r= Modulus $F=(a^2+b^2)^{n/2}$ and Θ= Argument F=n arctan(b/a)

On eliminating the n, one finds-

r=exp(αΘ) with α=[ln(a^2+b^2]/[2 arctan(b/a)]=constant

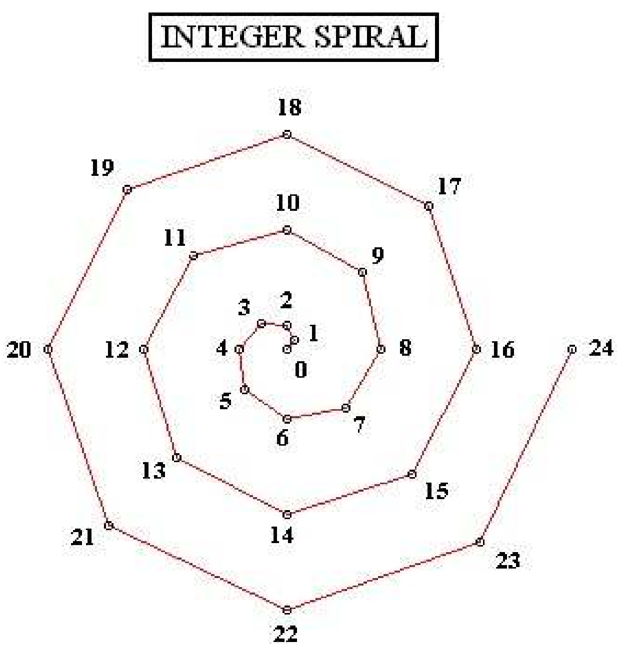
This figure represents the logarithmic Spiral of Bernoulli and it looks as follows for a=b=1



It was this figure which I used as a demonstration in our undergraduate complex analysis class which led to the discovery of the integer spiral-

$$r=n \quad \text{and} \quad \theta=n\pi/4$$

for all positive integers n. It produces the interesting picture-



in which all even integers lie along the x or y axis while all odd integers fall along the diagonals y=±x.