It is known that any complex number such as $N = a+ib$ and $M = c+id$ are located at unique points in the complex plane as shown in the following Argand Diagram-

The $i$ equals $\sqrt{-1}$. The absolute values of $N$ and $M$ are given as:

$$|N| = \sqrt{a^2 + b^2} \quad \text{and} \quad |M| = \sqrt{c^2 + d^2}$$

In looking at these definitions, it seems that one can introduce a third coordinate representing the known absolute value of the number. This implies we can designate the complex number $N$ by three unique coordinate points $[a, b, |N|]$ and $M$ by $[c, d, |M|]$. A 3D figure involving all of these coordinates for a complex number $N$ is the sphere defined by-

$$(x - a)^2 + (y - b)^2 + (z - |N|)^2 = |N|^2$$

This sphere, which lies directly above $[a, b.]$ in the complex plane, is tangent to this plane. The sphere has a radius of $R = |N|$.

We wish here to explore the properties of such spheres. Beginning with the complex number $N = -2i$, where $|N| = 2$, we have the sphere-

$$(x)^2 + (y + 2)^2 + (z - 2)^2 = 4$$
A plot of this 3D sphere lying directly above [0,-2] in the complex plane looks like this-

Note that the radius of this sphere has a unique value of two.

As a complex number increases in magnitude the absolute value of $\sqrt{a^2+b^2}$ increases so that the corresponding sphere has larger and larger radius. Neighboring spheres may collide with each other when the points [a,b] and [c,d] lie close enough to each other. The condition that two neighboring spheres just touch is easy to establish via a bit of geometrical manipulation. Using the properties of right triangles, we find that condition for the two spheres associated with $N= a+ib$ and $M= c+id$ to just touch is that-

$$16(a^2+b^2)(c^2+d^2) = [(a-c)^2 + (b-d)^2]^2$$

We can use this result to answer several questions which come to mind. First of all, we can ask if you take n equal sized spheres and lay them out in a circle, how far from the center of this circle will it be to the center of any of the spheres? You may replace the spheres with marbles or even coins to get the answer to this problem. Here is the answer. If you have n spheres, each sphere subtends a total angle of $2\pi/n$ radians relative to the circle center. Drawing an isosceles triangle formed by two sphere centers and the center of the circle at the same elevation produces the result –

$$\sin(\frac{\pi}{n}) = \frac{R}{a}$$

Here R is the sphere radius and ‘a’ the horizontal distance from a sphere center to the center of the ring. So if we make a ring of six spheres, marbles, or coins, we have $a=2R$. This means one has enough room to place a seventh sphere in the
Many of you will have played in your youth with the arrangement of six pennies to form a circle and will already have noticed that a seventh penny just fits into the center of the ring. The placement of four identical spheres of unit radius each placed in a ring can be represented by the complex numbers $z_1 = a, z_2 = ia, z_3 = -a$ and $z_4 = -ia$, Here we have $b = c = 0$ and $a = d$. Hence, by the geometry, we have-

$$2R = a\sqrt{2} \quad so \quad that \quad a = d = \sqrt{2}$$

A graph of these four spheres follows-

The next question we ask is what is the size of the largest marble which can be placed into the depression formed by three equal sized larger marbles arranged in a circular array such that the smaller central marble does not show its head above the three larger outer marbles? This question must be answered in two parts. First we note the distance between any two of the centers of the touching outer marbles is $2R$ and that the figure projected onto the complex plane by the three marble centers forms an equilateral triangle with side length $2R$. Thus the center projection onto the complex plane of any of the outer marbles and the origin will be $a = 2R/\sqrt{3}$. We next turn the marble array upside down and there see that the smaller inner marble of radius $r$ is flush with the plane as are the outer marbles. Applying a little geometry then says-

$$(r + R)^2 = (R - r)^2 + a^2 \quad which \quad means \quad r = \left(\frac{1}{3}\right)R$$
That is, the largest marble which may be placed into the central depression has a diameter of one-third of any of the outer marbles.

A related problem deals with the stacking of equal sized cannon balls. Here the base triangle remains as before but the fourth cannon-ball fitting into central depression will stick \(2/3\ R\) above the plane formed by the top of the three bottom balls. The equations defining the surfaces of the four cannon-balls will be-

\[
x^2 + \left( y - \frac{2}{\sqrt{3}} \right)^2 + (z - 1)^2 = 1
\]
\[
(x - 1)^2 + \left( y + \frac{1}{\sqrt{3}} \right)^2 + (z - 1)^2 = 1
\]
\[
(x + 1)^2 + \left( y + \frac{1}{\sqrt{3}} \right)^2 + (z - 1)^2 = 1
\]
\[
x^2 + y^2 + \left( z - \frac{8}{3} \right)^2 = 1
\]

A rendering of the four stacked cannon-balls using the operation implicitplot3d in our MAPLE program produces the following-

![Top View of Four Stacked Cannon-Balls](image)

When dealing with spheres of unequal radius just touching each other, there will generally be a limited number of values for which \(a, b, c,\) and \(d\) all remain integers. Let as demonstrate this for the sphere pairs governed by-
\[ z_1 = 1 + i \quad \text{and} \quad z_2 = c + id \]

Since we want the two corresponding spheres to be touching, we must have, from the earlier touching formula, that-

\[ 32(c^2 + d^2) = [(1 - c)^2 + (1 - d)^2]^2 \]

Performing a contour plot of this expression for a contour of zero, produces a solution in form of a closed epicycloid–like curve as shown-

The second sphere governed by \( z = c + id \) can lie at any point along this last figure, but a thorough search shows that the only sphere for which \( c \) and \( d \) are both integers is \( z = -1 - i \) corresponding to the sphere-

\[ (x + 1)^2 + (y + 1)^2 + (z - \sqrt{2})^2 = 2 \]

Although in this case only one all integer solution for \( z \) was found, in most cases one will encounter two or more possibilities. Let us demonstrate using a new seed sphere generated by \( z = 3 \). This sphere has radius \( R = 3 \) and sits directly above the point \( x = 3 \) and \( y = 0 \) in the complex plane. Here our touching sphere equation predicts a complex number \( z - c + id \) solution lying an a different epicycloid-like double loop as shown-.
In this case one finds three different spheres for which $z=c+id$ have all integer components. They are:

$$z = -3, \ z = 9-12i, \text{ and } z = 9-12i$$

One can generalize things from the last two examples and say that the seed − sphere generated by $z=a+ib$ predicts that a second touching sphere will have $c$ and $d$ required to lie on a double-looped epicycloid-like figure whose symmetry axis lies along a line in the $c$-$d$ plane at angle $\varphi=\arctan(d/c)$. Among the infinite number of possibilities there will most often be a few solutions for which both $c$ and $d$ are integers.

The spheres generated by $z=a+id$ have their centers located at $[a,b,|N|]$ and have a radius of $|N|$.

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