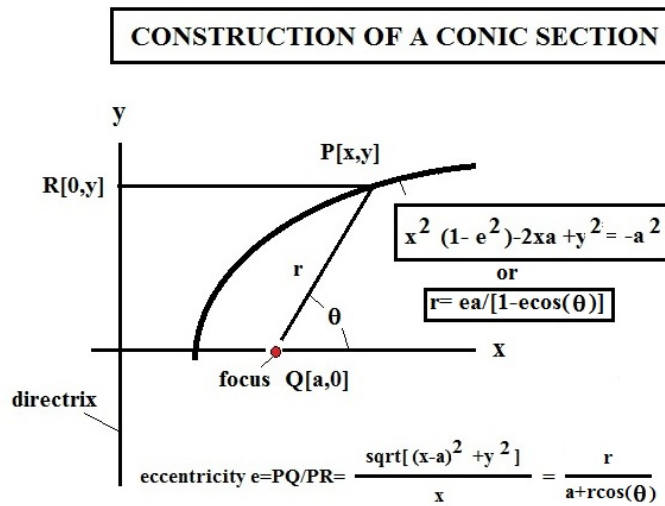# CONIC SECTIONS

One of the most important areas of analytic geometry involves the concept of conic sections. These represent 2d curves formed by looking at the intersection of a transparent cone by a plane tilted at various angles with respect to the cone axis. The resultant intersections can produce circles, ellipses, parabolas, and hyperbolas. Collectively they are referred to as conic sections. We want here to review their properties.

Our starting point is the following definition sketch-

The construction of a conic section starts with drawing a horizontal x axis and a vertical y axis termed the directrix. One next chooses a point Q(a,0] on the x axis termed the focus. If we then sketch a 2D curve represented by a heavy thick curve in the x-y plane and demand that this curve have the property that any point P(x,y) on it have the ratio r/x be a constant, we will have the definition of a conic section. Calling this ratio the eccentricity of the curve, we find-

$$e = \frac{r}{a + r\cos(\theta)} = \frac{\sqrt{(x-a)^2 + y^2}}{x}$$

On solving for r and x we get the polar coordinate and Cartesian versions of a conic section, namely,-

$$r = \frac{ea}{1 - e\cos(\theta)} \quad and \quad x^2(1 - e^2) - 2ax + y^2 = -a^2$$

Looking at the Cartesian version, one has an ellipse if e<1, a parabola if e=1, and a hyperbola when e>1. A circle is a special form of an ellipse with zero eccentricity (e=0).

We look at all three finite e cases starting with the ellipse. First one converts the Cartesian form of the conic section into the more recognizable form-

$$\frac{(x-b)^2}{(eb)^2} + \frac{y^2}{(e^2 ab)} = 1 \quad where \quad b = \frac{a}{(1-e^2)}$$

Ellipse: Here e<1 so that b>1. This means we have a standard shifted ellipse of the form
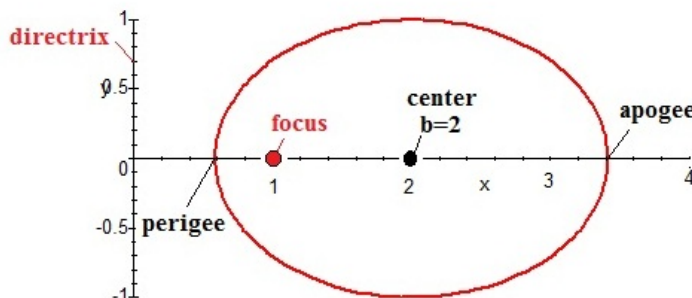
$$\frac{(x-b)^2}{A^2} + \frac{y^2}{B^2} = 1$$

which is centered on [b,0] and has a semi-major axis A=eb=ea/(1-e^2) and semi-minor axis
B =e sqrt(ab)= ea/sqrt(1-e^2). If we take the quotient sqrt(A^2-B^2)/A, we find this ratio equals-

$$\sqrt{1-(\frac{B}{A})} = e \ \ the \ eccentricity \ of \ the \ ellipse$$

Taking a=1 and e=1/sqrt(2) ,we have A=sqrt(2) , B=1, and b=2 to yield the graph-

ELLIPSE (x-2)^2+2y^2 =2 FOR a=1 AND e=1/sqrt(2)

The directrix remains the y axis and the focus of the ellipse is located at [1,0]. The point on the ellipse closest to the focus is the perigee and the point furthest away is termed the apogee. This ellipse has a second focus located at [3,0]. An

interesting property of ellipses is that the total distance from the first focus to any point on the ellipse and then the reflection onto the second focus is a constant equal to twice the semi-major axis A The semi-minor axis is given by the above expression as B=A sqrt(1-e^2). This equal time-travel property has found important applications in whispering galleries, hydrogen bomb triggers, and kidney stone removal. Elliptical trajectories of this type also have a very important application in connection with planetary trajectories about the sun. In that application one prefers the use of the following polar representation -

$$r = \frac{ea}{1 - e\cos(\theta)} \quad with \quad e < 1$$

From it one sees at once that the distance from the focus to the left intersection is the length to the perigee r_p=ea/(1+e) while the distance from the first focus to the right intersection at the apogee equals r_a=ea/(1-e). The sum r_p+r_a=2A.

Parabola: Here e=1 so that the conic section takes on the form of the shifted parabola-

$$(\frac{y^2}{2a}) = x - \frac{a}{2}$$

A graph of this parabola for a=2 looks like this-

GRAPH OF THE PARABOLA y^2=4(x-1) WITH a=2 AND e=1
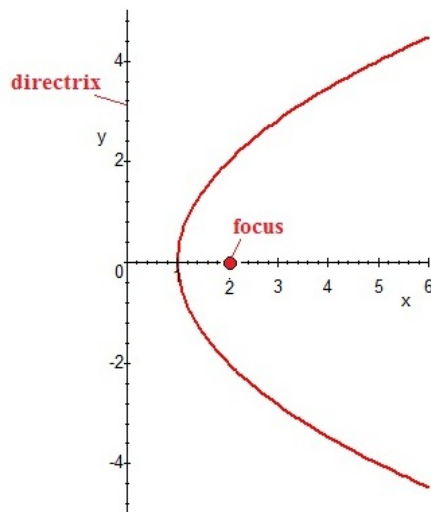
An important property of this conic is that all incoming light rays parallel to the x axis will focus at the focal point at [2,0]. This focusing principle lies behind all parabolic reflectors be they used for receivers such as reflector telescopes or sending out a parallel beam of light such as in search lights.

Hyperbola: Here e>1 so that b<0. So the conic section has the form of the hyperbola-

$$\frac{(x+|b|)^2}{(e|b|)^2} - \frac{y^2}{e^2 a|b|} = 1$$

A graph of this hyperbola for the special case of a=3 and e=2 follows-

GRAPH OF THE HYPERBOLA $3(x+1)^2 - y^2 = 12$ for a=3 and e=2
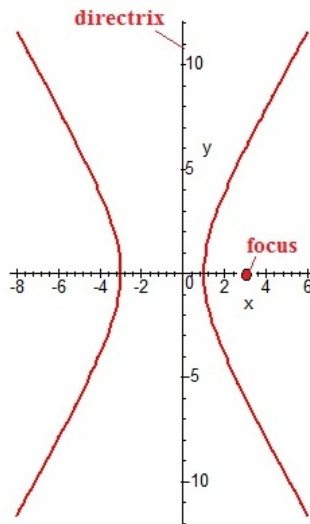
Note here that the hyperbola center is at x=-1 and y=0 while the focus for the right branch lies at [3,0]. We can calculate the curves perpendicular to these branches by noting they must have the slope-

$$\frac{dy}{dx} = -\frac{y}{3(x-1)}$$

and thus form the orthogonal curves-

$$y^3 = \frac{const.}{(x-1)}$$

, where the constant is adjusted to match a point on the hyperbola. In certain incompressible and inviscid 2D fluid flows one encounters flows having hyperbolic streamline shapes of this type.