ON DIFFERENCE EQUATIONS

A difference equation is one where a function F[n] which is defined only for integers n is related to the function F[n+a] where a is also an integer. A simple form of a difference equation is-

F[n+1]=F[n]+1 subject to F[1]=1

By inspection one has F[2]=1+1=2, F[3]=2+1=3, F[4]=1+3=4, etc. So the solution is F[n]=n. A slightly more complicated form is given by the equation-

F[n+1]=F[n]+(n+1) starting with F[1]=1

We find F[2]=3, F[3]=6, F[4]=10, etc. So it is clear that F[n]=n(n+1)/2. Hence F[5]=15 and F[6]=21. This last solution just represents the sum of the integers n=1 through n=n. That is-

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Going on, we consider the difference equation-

F[n+1]=F[n]+$(n+1)^2$ subject to F[1]=1

It yields F[2]=5, F[3]=14, F[4]=30. So we suspect that F[n] will have the form-

F[n]=An^3+Bn^2+Cn

We can use the evaluated values for n=1, 2, and 3 to get the matrix equation-

$$\begin{bmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 27 & 9 & 3 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 14 \end{bmatrix}$$

This solves as A= 1/3 , B=1/2 , and C= 1/6 . Hence the above difference equation has the general solution-

$$F[n]=\frac{1}{6}\left\{2n^3+3n^2+n\right\}$$

This result is equivalent to summing up the squares of the integers from 1 through n.

To get the sum of the integers to the integer power p of n, we have the difference equation-

$F[n+1]=F[n]+(n+1)^p$ subject to $F[1]=1$

This yields $F[2]=1+2^p$, $F[3]=1+2^p+3^p$, $F[4]= 1+2^p+3^p+4^p$, etc. Writing down enough of these lower values of F[n] will allow us to write that the sum-

$$1+2^p+3^p+......+n^p=\sum_{k=1}^{p+1}a_k n^k$$

,where the constants a_k are determined by solving a matrix equation with a (p+1) x (p+1) coefficient matrix.

A topic which can be well explained by a simple linear difference equation is the return on money given an interest rate of i.. One has the expression-

$F[n+1]=(1+i)F[n]$ subject to $F[0]=F_o$

, where i is the interest rate. Here we find the general solution-

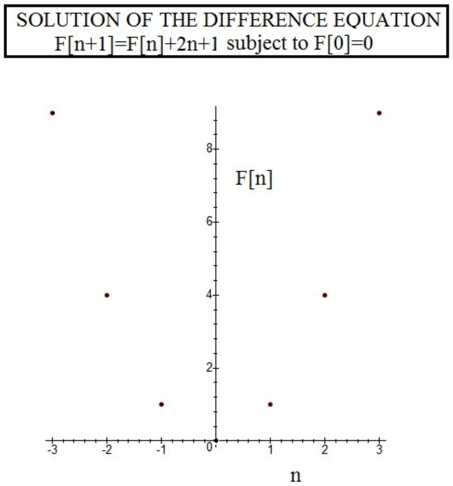
$$F[n]=F_o(1+i)^n$$

after n years. So to double one's money will take ln(2)=nln(1+i)≈ ni for low i . Since ln(2)=0.693147.., it will take approximately ten years to double ones money at an interest rate of 7%.

When dealing with difference equations it is often easier to reverse the procedure and go from a solution to the equation. This can be demonstrated by the solution $F[n]=n^2$. here we can write $F[n+1]=(n+1)^2=n^2 +2n +1$. Hence we have the difference equation-

F[n+1]=F[n}+2n+1 subject to F[0]=0

One recognizes that in this instance F[n] represents a parabola evaluated only at its integer points. A pointplot of the solution looks like this-



Partial sums of infinite series may also be expressed as finite difference equations. Consider the equation-

F[n+1]=F[n]+1/(n+1)! subject to F[0]=1

Here we find F[1]=2, F[2]=2.5, F[3]=8/3=2.666.., and F[4]=65/24=2.708.. one recognizes thatb these Fs are just the partial sums leading to F[∞]=exp(1)=2.718281828459045..The last is the base for the natural logarithm of numbers.

The Fibonacci Numbers are given by 1,2,3,5,8,13, etc and are generated by the difference equation-

F[n+2]=F[n+1]+F[n] subject to F[0]=1 and F[1]=2

One has F[2]=3,F[4]=5,F[5]=8, F[6]=13, and F[7[=21. asfirst noted by Johanes Kepler of astronomy fame, theb ratio F[n+1]/F[N] approaches a constant value of {sqrt(5)+1}/2 as n becomes infinite. This number represents the Golden Ratio φ= F[∞]=1.618033989.. .

The non-linear difference equation leading to the Golden Ratio is given by the non-linear difference equation-

$$F[n+1] = 1 + \frac{1}{F[n]} \quad subject \quad to \quad F[0] = 1$$

Has F[1]=2, F[2]=3/2, F[3]=5/3, and F[4]=8/5. The solution approaches F[∞]=φ. An interesting property of φ that it satisfies the following continued fraction-

$$\varphi = !+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+...}}}}$$

as is easily established by noting {φ-1)(φ+1)=φ^2-1=φ.

A difference equation close in appearance to the equation leading to the Golden Ratio is

$$F[n+1] = 1 + \frac{1}{\{1+F[n]\}} \quad subject \quad to \quad F[0]=0$$

Here we get F[1]=2, F[2]=4/3, F[3]=10/7, F[4]=24/17, F[5]=58/41. As n gets large the value of F[n] approaches the square root of two. The equation can thus be thought of as an iteration formula for sqrt(2).

Another highly non-linear difference equation is-

F[n+1]=F[n]+cos(F[n]){cos(F[n])-sin(F[n])} subject to F[0]=1

Its solutions read F[1]=0.837…, F[2]=0.7881…, F[3]=0.78540,,, F[4]=0.785398163

If we compare the F[4[term with π/4=0.785398163…, we see that here we have an iteration for π given by –

$$\pi = \frac{\lim}{n \to \infty}\{4F[n]\}$$

Already at n=7 one finds a 78 digit accurate result for π. For a derivation of the above iteration formula see the discussion in our article on iterating the Taylor series for arctan .The article is found on our MATHFUNC page dated Dec.2012.

Many special algebraic equations y=F(x) have integer solutions for special cases. Equations of that type are classified as Diophantine equations. They have been studied for nearly 2000 years. Included among this group is the Brahmagupta equation-

$[y(x)]^2=1+C\,x^2$ where C is a specified integer

where one specifies the constant C beforehand and then tries to find integer solutions for y corresponding to an integer x=n. In difference equation form the equation reads-

$\{F[n+1]\}^2=\{F[n]\}^2+C(2n+1)$ subject to F[0]=1

One finds $F[1]^2=(1+C)$, $F[2]^2=1+4C$, $F[3]^2=1+9C$. It is clear that integer solutions for F[n] only if the right side of $F[n]^2$represents the square of an integer. Thus if C=3 then [x,y] =[n,F[n]]=[1,2],[[4,7],[15,26],[56,97[,[209,362],[780,1351],[2911,5042], and so on are the only integer solutions. Note for large integers the ratio F[n]/n goes as sqrt(3)=1.732 and the ratio of (n+1)/n goes as 2+sqrt(3)=3.73205. Sometimes this equation will have no integer solutions at all while at other times only one integer solution becomes possible. Certainly when C equals the square of an integer the factor 1 in the equation solution makes any integer solution impossible. So, for example $F[n]^2=1+169n^2$ will never yield an integer solution other than the trivial result [1,1].

The secret to being able to factor large semi=primes is that one find the value of n which Makes the solution $F[n]=sqrt(An^2+Bn+C)$ an integer for specified integer values of A,B, and C.values. That is, one is looking for integer solutions of F[n] satisfying the non-linear difference equation –

$$F[n+1]^2 = F[n]^2 + A(1+2n) + B \quad subject \quad to \quad F[0] = sqrt(C)$$

The only integer solutions found for the special case of A=4, B=3, and C=6 are-

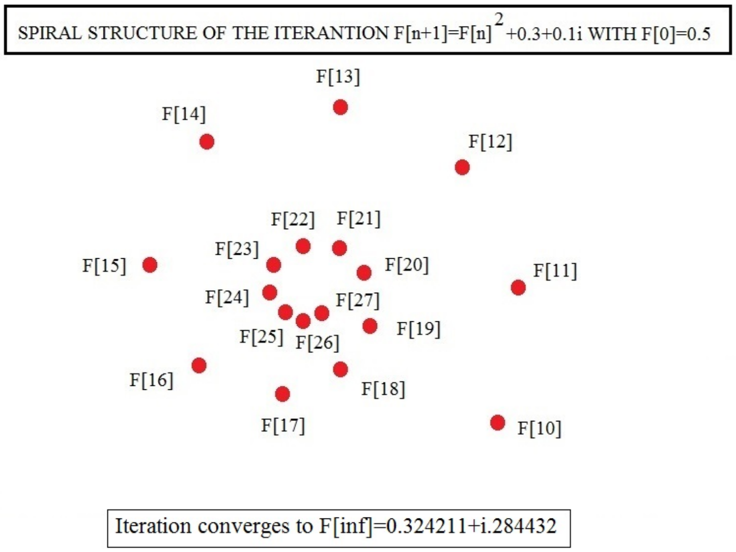
[n,F[n]]=[5,11] and [-2,4]

Nothing further in the range -400<n<600.

Finally let us look at the complex finite difference equation-

$F[n+1]= F[n]^2 +0.3+i0.1$ subjected to F[0]=0.5

Here we find F[1]=0.55+0.1i and F[2]=0.5925+0.210i. The solutions converge to a point near 0.324211+i0.284431 in the complex plane. A picture of the interesting inward moving spiral representing these solutions follows-



I leave it for the reader to show that the exact form for the solution F[∞] is given by-

$$F[\infty]=(\frac{1}{2})\{1+\sqrt{\frac{1+2i}{5}}\}$$

Also the F[∞] result is here independent of the starting condition F[0].

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