SOLUTION OF THE DIOPHANTINE EQUATION

\[ x_1^p + x_2^p + \ldots + x_{p+1}^p = x_{p+2}^p \]

Perhaps the most famous algebraic equations involving the sum of variables \( x_n \) taken to the \( p \)th power is the one shown in the above title. When the variables \( x_n \) are integers it also represents a classic Diophantine Equation. It is the purpose of this note to study this equation in more detail, restricting our attention to strictly integer solutions.

The simplest form of such an equation is the linear Diophantine form-

\[ x_1 + x_2 = x_3 \]

where \( p=1 \). It has an infinite number of integer solutions starting with \( 1+2=3 \). Other possibilities are \( 3+7=10 \), \( 27+53=80 \) etc. There is really nothing more interesting to say about this form.

However, starting with \( p=2 \), one gets more interesting results. The equation now reads-

\[ x_1^2 + x_2^2 = x_3^2 \]

This is just the Pythagorean Theorem with \( x_3 > x_2 > x_1 \). We can represent its integer solutions via the Pythagorean Triplet \([x_1,x_2,x_3]\). One of the simplest ways to find these integer triplets is to first assume \( x_2+1 = x_3 \). This yields-

\[ [x_1, x_2, x_3] = [\sqrt{2x_2 + 1}, x_2, x_2 + 1] \]

Searching using \( x_2 \) equal 2 through 100, produces the triplets-

\[ [3,4,5], [5,12,13], [7,24,25], [9,40,41], [11,60,61], [13,84,85] \]

These are called base triplets since their elements do not have a common divisor except 1. One can always generate additional triplets by multiplying the elements in a base triplet by any positive integer \( n \). Thus \([51,68,85]\) would be a possible non-base triplet.

Going on to the second possibility that \( x_3 = x_2 + 2 \) we get the additional triplets in the range \( x_2 \) equal 5 through 100 of-

\[ [12,35,37], [16,63,65], [20,99,101] \]

We have here reduced things to base triplets and do not repeat what has already been found in the previous case. Going on to the possibility of \( x_3 = x_2 + 3 \) in the same \( x_2 \) range, we find no new triplets which do not reduce to one of the earlier base triplets. The same is true for \( x_3 = x_2 + 4 \). We can thus say that all base Pythagorean Triplets have-
\[ \sqrt{2x^2+1}, x_2, x_2+1 \] or \[ 2\sqrt{x_2+1}, x_2, x_3+2 \]

A physical interpretation of these triplets is that they represent the sides of a right triangle with the hypotenuse equal to \( x_3 \). Or even simpler, the sum of two squares of sides \( x_1 \) and \( x_2 \) equal a third square of side-length \( x_3 \). In my wood workshop I often make use of the 3-4-5 right triangle to construct an accurate right angle. I point out there is often the possibility of having more than two terms on the left hand side for the \( p=2 \) case. For example, we have-

\[ 2^2 + 3^2 + 6^2 = 7^2 \]

About 400 years ago the famous French lawyer and mathematician Pierre Fermat (1601-1665) was playing around with the Pythagorean Theorem. He tried to extend the theorem to powers higher than \( p=2 \) but was unsuccessful. It led him to his famous Last Theorem that-

\[ x_1^p + x_2^p \neq x_3^p \] for any integer power greater than \( p=2 \)

It took until recently to show that the theorem was indeed correct (see A.Wiles).

What apparently never occurred to him and others is that \( p=3 \) solutions are possible for the four term Diophantine Equation-

\[ x_1^3 + x_2^3 + x_3^3 = x_4^3 \]

We can show this by making use of the following power table-

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<th>( n^1 )</th>
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</table>
and looking at the elements in the third column. Note that the elements in this column alternate between even and odd as they also do for other integer ps. To search for integer solutions we first let \( x_3 + 1 = x_4 \). This produces the result:

\[
x_1^3 + x_2^3 = 3x_3(x_3 + 1) + 1
\]

Trying \( x_3 = 5 \) produces \( x_1^3 + x_2^3 = 91 \). But from the table this says \( x_1 = 3 \) and \( x_2 = 4 \). So we arrive at the quartic form \([3,4,5,6]\) from which we have the identity-

\[
3^3 + 4^3 + 5^3 = 6^3
\]

If we take \( x_3 = 8 \) one finds an additional quartic \([1,6,8,9]\) producing the identity-

\[
1^3 + 6^3 + 8^3 = 9^3
\]

We are not certain whether there exist additional base quartics which sum some of the third column elements. We can however create an infinite number of non-base quartics by multiplying each element in the above two quartics by an integer \( n \). Thus we have among an infinite number of other quartics the identity-

\[
729^3 + 972^3 + 1215^3 = 1458^3
\]

Also note that more than three terms on the left of the \( p=3 \) Diophantine equation are possible. We have, for example,-

\[
1^3 + 5^3 + 7^3 + 12^3 = 13^3
\]

Drawing the analogy between Pythagorean Triplets and Fermat’s Theorem, we can state that it is very likely that-

\[
a^p + b^p + c^p \neq d^p \quad \text{for any} \quad p \geq 4
\]

Note that a physical interpretation of the quartic \([a,b,c,d]\) is that it represents the volume of three cubes of side-length \( x_1, x_2, \) and \( x_3 \) each equal to a larger cube of side-length \( x_4 \).

We next come to the 4th column of the above power table where \( p=4 \). Does it lend itself to an integer solution of-

\[
x_1^4 + x_2^4 + x_3^4 + x_4^4 = x_5^4
\]
We try to find such a quintic \([a,b,c,d,e]\) by starting with the assumption that \(e=d+1\). So we get:

\[a^4 + b^4 + c^4 = 4d^3 + 6d^2 + 4d + 1\]

This yields no all integer solutions for \(d\) from 6 through 20 suggesting there is no such solution for any combinations of integers taken to the 4th power. Hence it is very likely that no quartics or higher power sums with all integer solutions exist.

Finally we leave you with the following inequality:

\[\sum_{n=1}^{p-1} (x_n)^p \neq (x_p)^p \text{ for } p \geq 3\]

It contains as a special case \((p=3)\) the Fermat Theorem. A proof should be possible using elliptic curves.

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