EULER’S CONSTANT

An important constant associated with the prolific nineteenth century Swiss mathematician Leonard Euler (1707-1783) is-

$$\gamma = 0.5772156649015$$

It is also known as the Euler-Mascheroni constant and arises in a variety of different areas including definite integrals, summation of series, and as part of the solution of certain differential equations. Our purpose here is to derive the value of this constant and then discuss its properties in detail.

Our starting point is to look at the area underneath the parabola $y=1/x$ extending from $x=1$ to $x=m$. This equals-

$$G(m) = \int_{x=1}^{m} \frac{dx}{x} = \ln(m)$$

This integral clearly blows up as $m \to \infty$. Next one looks a the area beneath a descending step function. This step-function can be written as –

$$T(m) = 1 + \sum_{n=2}^{m} \left[ \frac{H(n-1) - H(n)}{(n-1)} \right] = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{(m-1)}$$

Here $H(n)$ is the Heaviside Function. For $m \to \infty$ we recognize this series to be the harmonic series which also becomes unbounded as $m$ heads to infinity. Now what Euler noticed is that despite $G(x)$ and $T(x)$ becoming infinite as $x$ goes to infinity, the difference does not but rather approaches a constant known as Euler’s constant. The following graph will make this more clear-
The sum of the grey shaded areas seems to approach a constant value. To find out the area limit, we look at the function:

\[ K(x) = \sum_{x=1}^{\infty} \frac{1}{x} - \ln(x) \]

Working out a few values by hand starting with \( K(2) \), we find:

\[ \begin{align*}
K(2) &= 1 - \ln(2) = 0.3068.. \\
K(3) &= \frac{3}{2} - \ln(3) = 0.4013.. \\
K(4) &= \frac{11}{6} - \ln(4) = 0.4470.. 
\end{align*} \]

This sequence appears to converge but does so very slowly. With aid of our computer we find the additional values in the sequence:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( K(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.52638316…</td>
</tr>
<tr>
<td>100</td>
<td>0.57220733…</td>
</tr>
<tr>
<td>1000</td>
<td>0.57671558…</td>
</tr>
<tr>
<td>10000</td>
<td>0.57716566…</td>
</tr>
<tr>
<td>100000</td>
<td>0.57721066…</td>
</tr>
<tr>
<td>1000000</td>
<td>0.57721516…</td>
</tr>
<tr>
<td>10000000</td>
<td>0.57721562..</td>
</tr>
</tbody>
</table>
These results compare with the exact value of the Euler constant given as:

\[ \gamma = K(\infty) = 0.5772156649015328606065120900824024310422\ldots \]

The sequence \( K(x) \) is thus seen to be very slowly convergent by noting that even taking ten million terms the constant accuracy is still only good to seven places. The definition of \( \gamma \) follows as:

\[
\gamma = \lim_{n \to \infty} \left\{ \frac{e^{-1}}{n} \sum_{k=1}^{n} \frac{1}{k} - \int_{x=1}^{n} dx \right\} = \lim_{n \to \infty} \left\{ \frac{1}{n} \left( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{(n-1)} \right) - \ln(n) \right\}
\]

There are some obvious results stemming from the definition of \( \gamma \). We have, for example, that:

The sum of the reciprocal integers up to and including \( 1/(m-1) \) is equal:

\[
\sum_{k=1}^{m-1} \frac{1}{k} = K(m) + \ln(m) = \Psi(m) + \gamma
\]

The last part of this equality is predicted by my computer. Here \( \Psi(m) \) is the Psi function (also known as the digamma function). The standard definition of the digamma function is:

\[
\Psi(z) = \frac{d[\Gamma(z)]/dz}{\Gamma(z)}
\]

where \( \Gamma(z) \) is the standard gamma function. A computer evaluation of \( \Psi(n) \) yields \( \Psi(1) = -\gamma, \Psi(2) = 1 - \gamma, \Psi(3) = (3/2) - \gamma \) and \( \Psi(4) = (11/6) - \gamma \). From these we obtain at once the simple recurrence relation:

\[
\Psi(z + 1) = \Psi(z) + \frac{1}{z}
\]

This relation also follows by differentiating the gamma function recurrence relation \( \Gamma(z+1) = z\Gamma(z) \). We have:

\[
\Psi(z + 1) - \Psi(z) = \frac{\Gamma(z)}{\Gamma(z + 1)} \left[ 1 + z\Psi(z) \right] - \Psi(z) = \frac{\Gamma(z)}{\Gamma(z + 1)} = \frac{1}{z}
\]

We can use the above to show that the sum of the reciprocals of the first 99 integers exactly equals to:

\[
\sum_{k=1}^{99} \frac{1}{k} = \Psi(100) + \gamma = 5.177377517639620\ldots
\]
The shaded area shown in the above graph from x=1 to x=100 equals K(100)=5.17735176-ln(100)= 0.572207332…. 

What needs to still be shown is that \( \Psi(1) = -\gamma \). To find this, we write down the identity relating \( K(m) \) to \( \Psi(m) \) as-

\[
K(m) = \Psi(m) + \gamma - \ln(m)
\]

Then setting \( m = 1 \) yields-

\[
\Psi(1) = -\gamma + \ln(1) + K(0) = -\gamma
\]

Certain definite integrals can also be related to Euler’s constant. For example, the integral definition of the gamma function \( \Gamma(z) \) reads-

\[
\Gamma(z) = \int_{t=0}^{\infty} t^{z-1} \exp(-t) dt
\]

The first derivative yields-

\[
\frac{d\Gamma(z)}{dz} = \int_{t=0}^{\infty} \ln(t) t^{z-1} \exp(-t) dt
\]

Taking the quotient of these last two expressions and then setting \( z=1 \) yields-

\[
-\gamma = \int_{t=0}^{\infty} \ln(t) \exp(-t) dt
\]

Also one has-

\[
1 - \gamma = \int_{t=0}^{\infty} t \ln(t) \exp(-t) dt \quad \text{and} \quad 3 - 2\gamma = \int_{t=0}^{\infty} t^2 \ln(t) \exp(-t) dt
\]

The most general form of this type of integral solves as-

\[
\int_{t=0}^{\infty} t^{n-1} \ln(t) \exp(-t) dt = \Psi(n) \Gamma(n)
\]

On applying the recurrence relation between \( \Psi(z+1) \) and \( \Psi(z) \) multiple times produces the identity-
\[ \Psi(z + m) = \Psi(z) + 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots + \frac{1}{z^{m-1}} \]

Choosing \( z = 1 \), this produces-

\[ \Psi(1 + m) = -\gamma + \sum_{k=0}^{m-1} \frac{1}{(1 + k)} = -\gamma + \sum_{n=1}^{m} \frac{1}{n} \]

Another integral which is expressible in terms of Euler’s constant is-

\[ \int_0^\infty \exp(-t) \left( \frac{1}{1 - \exp(-t)} - \frac{1}{t} \right) dt = \gamma \]

Note that the integrand \( \exp(-t)/t \) would lead to an infinite integral. This singularity is removed by the combination with the \( \exp(-t)/(1-\exp(-t)) \) term. Some other integrals involving \( \gamma \) are-

\[ \int_0^\infty \exp(-t) \ln(t) \frac{\sqrt{t}}{\sqrt{t}} \, dt = -\sqrt{\pi} \{ \gamma + \ln(4) \} \quad \text{and} \quad \int_0^\infty \exp(-t^2) \ln(t) = \frac{\sqrt{\pi}}{4} \{ \gamma + \ln(4) \} \]

Although the above offer no simple approach for rapidly evaluating \( \gamma \) to high accuracy using just a few terms in an expansion, people have now extended the accuracy of the Euler constant to well over ten billion digital places using iteration approaches. One procedure is to look at the identity-

\[ \gamma = \lim_{x \to 0} \left\{ \frac{1}{x} \left( 1 - \Gamma(x) \right) \right\} \]

After using the identity \( x\Gamma(x) = \Gamma(x+1) \) and replacing \( x \) by \( 1/z \), we find-

\[ \gamma = \lim_{z \to \infty} \frac{z}{\infty} \left\{ 1 - \Gamma\left( \frac{z + 1}{z} \right) \right\} \]

This allows us to introduce the new function-

\[ V(z) = z[1 - \Gamma(1 + \frac{1}{z})] \]
which is easy to evaluate for large $z$. As $z$ approaches infinity $V(z) \to \gamma$. it yields essentially the same slowly convergence as the $K(x)$ function defined earlier. However it has the advantage that there is only a single function $\Gamma(1+1/z)$ involved. Remember that-

$$\Gamma(1 + \frac{1}{z}) = \int_{t=0}^{\infty} t^{(1/z)} \exp(-t) dt$$

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