RAPID EVALUATION OF THE NUMBER FRACTION
AND SIGMA FUNCTION

INTRODUCTION:

It is well known that any positive integer can be represented by a finite number of primes taken to specified integer powers. This equality between a number \( N \) and its prime products is found in most mathematical programs such as MAPLE and MATHEMATICA. Thus, for example, \( N=37289 \) can be written as-

\[
N = 37289175488 = 2^6 \cdot (463) \cdot (1258409)
\]

where 2, 463 and 1258409 are primes. One can use such products to quickly calculate the values of these two important number theory point functions. The first of these is the long known sigma function \( \sigma(N) \) which represents the sum of all the divisors of a number \( N \). The second is the number fraction discovered by us earlier in the decade and defined as-

\[
f(N) = \frac{\sigma(N) - N - 1}{N}
\]

It has the property that \( f(N)=0 \) whenever \( N \) is a prime. It has come in very handy in the discussion of super-composites, twin primes, and the fact that all primes five or greater must have the form \( 6n+1 \) or \( 6n-1 \). The sigma and \( f(N) \) functions are related to each other via the identity-

\[
\sigma(N) = N \{ f(N) + 1 \} + 1
\]

Thus when \( N \) is a prime, \( f(N)=0 \) and \( \sigma(N) =N+1 \). The sigma function always appears as an integer which can become quite large as \( N \) gets large. The number fraction on the other hand is expressed as the quotient of two integers and usually has relatively low values even as \( N \) goes toward infinity. A minimum evaluation effort is usually achieved by evaluating \( f(N) \) first using some generic formulas, followed by the finding \( \sigma(N) \) using the above identity.

It is our purpose here to investigate the functions \( f(N) \) and \( \sigma(N) \) in greater detail.

DERIVING GENERIC FORMULAS FOR \( f(N) \):

As already stated, any number can be represented as the product of a finite number of primes taken to specified powers. Usually the number of primes required to represent the
number N is small so that it becomes useful to have available some generic forms for N into which specific primes can be substituted. Let us begin with the small number N=3·5. By definition we here have- 

\[ N = 15 \quad \text{so} \quad f(N) = \frac{3 + 5}{15} = \frac{8}{15} \]

Replacing the primes 3 and 5 by the generic forms p and q, we have the generic formula-

\[ f(pq) = \frac{p + q}{pq} \]

valid for any two primes p and q not equal to each other. So if N=61·101=6161, we have at once that \( f(6161) = 162/6161 \) and \( \sigma(6161) = 6324 \).

For \( f(3·5·7) \) we get \( (3+5+7+15+21+35)/105 \). So on replacing 3 by p, 5 by q and 7 by r, we find the generic formula-

\[ f(pqr) = \frac{p + q + r + pq + pr + qr}{pqr} \]

Using the same approach we also find the formulas-

\[ f(p^2q) = \frac{p + p^2 + q + pq}{p^2q} \]

\[ f(p^2q^2) = \frac{p + p^2 + q + q^2 + pq + p^2q + pq^2}{p^2q^2} \]

\[ f(p^3q) = \frac{p + q + p^2 + pq}{p^3q} \]

We can also derive additional generic formulas. But this will not be necessary unless the factor of N contains additional products of p, q, r….

**Specific Evaluations of f(N) and \( \sigma(N) \):**

Let us now take some specific numbers N. Starting with N=3791=17·223, we find using the formula \( f(pq) = (p+q)/pq \), that-
Next we take the larger seven digit number $N=2185083$. Here we find-

$$N=2185083=3^3 \cdot 80929, \quad f(N)=1052116/2185083, \quad \sigma(N)=3237200$$

To get this result we first used the above generic version for $f(p^3q)$ and then substituted $p=3$ and $q=80929$ into it.

As a final evaluation consider the number $N=174636000$. Here we have-

$$N = 174636000 = 2^5 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11$$

to produce

$$f(N)=(638768591)/(174636000)=3.657714 \quad \text{and} \quad \sigma(N)=813404592$$

This time we did not construct a generic formula containing five different primes as this would have been a rather lengthy task. Instead we relied on the value of sigma(N) as given by our PC and then worked backwards to find $f(N)$. Note here that $f(N)$ is an interesting number since it towers above its neighbors as shown in the following $N$ versus $f(N)$ graph-

Such a number can be designated as a super-composite (or rich number). Sometimes the immediate neighborhood may have primes. That is $f(N\pm1)$ sometimes will vanish. A large semi-prime $N=pq$ usually has values lying very close to zero. Super-composites usually occur when the number N consists of products of small integer primes taken to
high powers. Thus the number \( N=2^{12} \cdot 3^3 = 995328 \) is another such super-composite. Its number fraction equals \( f(N) = 1.995518060 \) and its immediate neighborhood at \( N \pm 1 \) produces \( f(N \pm 1) = 0 \). Hence this is an example of a super-composite sandwiched between two prime numbers \( N=995328 \pm 1 \).

CONCLUDING REMARKS:

We have shown that values of \( f(N) \) and \( \sigma(N) \) can readily be calculated for large \( N \) following from the fact that any number \( N \) can be represented as the product of a low number of primes taken to specified powers. Some generic formulas are derived for obtaining \( f(N) \). The values for \( \sigma(N) \) follow directly from these number fraction results. Knowing the values of \( f(N) \) over a range of \( N \) allows one to quickly spot primes, twin-primes, and super-composite numbers.

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