EVALUATION OF THE FUNCTION $F(m) = \sum_{n=1}^{\infty} \left[ \frac{n^m}{2^n} \right]$

In a recent newspaper article (St.Petersburg Times, Sunday Oct.19, 2008) there appeared a story about a high school student with mathematical skills with an interest in attending college as a math major. In one of the pictures he is shown working out the sum of the series-

$$F(2) = \sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \frac{25}{32} + \frac{36}{64} + \frac{49}{128} + ... = 6$$

by a re-grouping procedure based on the simpler sums-

$$F(0) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \quad \text{and} \quad F(1) = \sum_{n=1}^{\infty} \frac{n}{2^n} = F(0) + \frac{1}{2} F(1) = 2F(0) = 2$$

Following a similar approach it should be possible to extend such summations to the more general function-

$$F(m) = \sum_{n=1}^{\infty} \frac{n^m}{2^n} = \frac{1}{2} + \frac{2^m}{4} + \frac{3^m}{8} + \frac{4^m}{16} + ...$$

One would expect the sums to go as-

$$F(m) = \sum_{k=0}^{m-1} A_k F(k) = A_0 F(0) + A_1 F(1) + A_2 F(2) + ... + A_{m-1} F(m-1)$$

We have carried out such evaluations with aid of our computer to find-
\[ F(1) = 2F(0) = 2 \]
\[ F(2) = 3F(1) = 6 \]
\[ F(3) = 4F(2) + F(1) = 26 \]
\[ F(4) = 5F(3) + 3F(2) + F(1) = 150 \]
\[ F(5) = 7F(4) + F(3) + F(2) = 1082 \]
\[ F(6) = 8F(5) + 4F(4) + 4F(3) + F(2) = 9366 \]

There appears to be no obvious pattern in the numbers \( A_k \) or in the values of the sums found for \( F(m) \) other than that the value of \( F(m) \) increases with increasing \( m \) and that the sums are all even integers.

This suggests that an alternate route must be taken to find a general pattern for \( F(m) \) and to find a simpler algorithm for finding its values. Let us do this. Recall from an earlier discussion that it is possible to convert infinite series to integrals via Laplace transforms. The procedure says that-

\[
\sum_{n=1}^{\infty} H(n) = \int_{0}^{\infty} \frac{h(t)}{\exp(t) - 1} dt
\]

where \( H(n) \) is the Laplace transform of \( h(t) \) with \( n \) replaced by \( s \). Using our MAPLE computer program one can show that-

\[
invlaplace\left(\frac{s^m}{2^s}\right) = Heaviside(t - \ln(2))Dirac(t, t - \ln(2))
\]

where \( Dirac(m, t - \ln(2)) \) refers to the \( m \)th derivative of the Dirac Delta Function. One can thus write that-

\[
F(m) = \int_{t=\ln(2)}^{\infty} \frac{d^m Dirac(t - \ln(2))}{\exp(t) - 1} dt
\]

Next we note by integration by parts that for any continuous function \( f(t) \) one has-

\[
\int_{t=-\infty}^{\infty} f(t)Dirac(m, t - a)dt = (-1)^m \frac{d^m f(t)}{dt^m} \bigg|_{t=a}
\]
so that-

\[ F(m) = (-1)^m \frac{d^m}{dt^m} \left[ \frac{1}{\exp(t) - 1} \right] \bigg|_{t = \ln(2)} \]

This is an extremely simple algorithm for finding the sum of the series \( F(m) \). It leads to the following results-

\[
\begin{align*}
F(1) &= 1! \cdot 2^1 = 2 \\
F(2) &= -[1] \cdot 1! \cdot 2^1 + 2! \cdot 2^2 = 6 \\
F(3) &= 1 \cdot 2^1 - [3] \cdot 2! \cdot 2^2 + 3! \cdot 2^3 = 26 \\
F(4) &= -1 \cdot 2^1 + [7] \cdot 2! \cdot 2^2 - [6] \cdot 3! \cdot 2^3 + 4! \cdot 2^4 = 150 \\
F(5) &= 1 \cdot 2^1 - [15] \cdot 2! \cdot 2^2 + [25] \cdot 3! \cdot 2^3 - [10] \cdot 4! \cdot 2^4 + 5! \cdot 2^5 = 1082 \\
F(6) &= -1 \cdot 2^1 + [31] \cdot 2! \cdot 2^2 - [90] \cdot 3! \cdot 2^3 + [65] \cdot 4! \cdot 2^4 - [15] \cdot 5! \cdot 2^5 + 6! \cdot 2^6 = 9366
\end{align*}
\]

This time one notices a definite pattern. The last term in each of the sums equals \( m! \cdot 2^m \), and the second to the last as \(-[m(m-1)/2](m-1)2^{m-1}\). The terms further to the left become more cumbersome before they again simplify yielding \( 2! \cdot 2^2 (-1)^m (2^{(m-1)-1}) \) and \( 1! \cdot 2^1 (-1)(m+1) \) for the first two terms to the right of the equality signs. In general one has-

\[
F(m) = \sum_{n=1}^{\infty} \frac{m^n}{2^n} = m! \cdot 2^m \left\{1 - \frac{(m-1)}{2} + \frac{(m-2)(3m-5)}{96} - \frac{(m-2)(m-3)^2}{384} + O(m^4)\right\}
\]

So for \( m=3 \) it follows that-

\[ F(3) = 3! \cdot 2^3 - 2! \cdot 2^2 [3] + 1! \cdot 2^1 [1] = 26 \]

and for \( m=4 \) we have-

\[ F(4) = 4! \cdot 2^4 [1 - \frac{3}{2} + \frac{2(7)}{96} - \frac{2(1)}{384}] = 150 \]

One can also construct a Pascal like triangle as shown-
and then can read the results for various F(m)s directly. Take the case m=8. It yields-

\[ F(8) = -[1] \cdot 1!2^1 + [127] \cdot 2!2^2 - [966] \cdot 3!2^3 + [1701] \cdot 4!2^4 - [1050] \cdot 5!2^5 \\
\quad + [266] \cdot 6!2^6 - [28] \cdot 7!2^7 + [1] \cdot 8!2^8 = 1,091,670 \]

To demonstrate the power of the above derivative algorithm for larger value of m, look at the case F(32). It takes only a fraction of a second to yield the following 41 digit result-

\[ F(32) = 47090308171469793298368981274289710952790 \]

Note again that the sum is an even number.

Finally let us briefly mention how it is possible to obtain approximate values for infinite series such as the present one. One needs only to replace the integer values in the infinite sum by the variable x and then write-

\[ F(m) = \sum_{n=1}^{\infty} \frac{n^m}{2^n} \approx \int_{x=0}^{\infty} x^m dx = \int_{x=0}^{\infty} x^m \exp[-x \ln(2)] dx = \Gamma(m+1)/[\ln(2)]^{m+1} \]

where the last integral has been evaluated exactly for this case by recognizing that it is just the Laplace transform of x^m with s replaced by ln(2). To see how good this approximation is, we find, for example, that \( \Gamma(3)/[\ln(2)]^3 = 6.0055614 \ldots \)
\[ \Gamma(7)/[\ln(2)]^7 = 9366.002503. \] These values should be compared to the exact sums \( F(2) = 6 \) and \( F(6) = 9366 \) given above. Another advantage of such integral approximations is that one can see at what approximate value the terms in the series reach their maximum. In the present case a simple differentiation \( x^m/2^x \) shows that the maximum value of the terms in the present series occurs near \( x = m/\ln(2) \) and has an approximate value of \( V = [m/\ln(2)]^{m/2} [m/\ln(2)] \). For \( m = 6 \) the maximum occurs near \( n = 8.656 \) and has a value \( V = 1042.768 \) as is nicely confirmed by the following graph-

![Graph showing maximum value](image)

Oct. 23, 2008

**Note added June 22, 2009**

I recently received an email from Michael Rodeman (the student written up in the St. Petersburg Times Oct. 2008) who tells me he will be attending the University of Florida starting this fall as a math major. He is presently working on the generalized sum-

\[
\sum_{n=1}^{\infty} \frac{n^a}{b^n} = \frac{1}{b} + \frac{2^a}{b^2} + \frac{3^a}{b^3} + 
\]

via his regrouping approach. One can also carry out the summation using the above derived differentiation approach based on Laplace transforms. One finds that-
\[ S(a,b) = \sum_{n=1}^{\infty} \frac{n^a}{b^n} = (-1)^a \frac{d^a}{dx^a} \left[ \frac{1}{\exp(x) - 1} \right]_{x=\ln(b)} \]

which produces the solution-

\[ S(a,b) = \sum_{n=1}^{n=a} \frac{(-1)^{a+1-n} K(a,n)b^{a+1-n}(b-1)^{n-1}}{(b-1)^{a+1}} \]

where \( K(a,n) \) are coefficients arising in carrying out the indicated differentiations. The values of these coefficients can best be presented in table form as follows:

<table>
<thead>
<tr>
<th></th>
<th>K(a,1)</th>
<th>K(a,2)</th>
<th>K(a,3)</th>
<th>K(a,4)</th>
<th>K(a,5)</th>
<th>K(a,6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a=1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a=2</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a=3</td>
<td>6</td>
<td>6</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a=4</td>
<td>24</td>
<td>36</td>
<td>14</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a=5</td>
<td>120</td>
<td>240</td>
<td>150</td>
<td>30</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>a=6</td>
<td>720</td>
<td>1800</td>
<td>1560</td>
<td>540</td>
<td>62</td>
<td>1</td>
</tr>
<tr>
<td>a=a</td>
<td>a!</td>
<td>a!(a-1)/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thus, for example,

\[ S(2,2) = \sum_{n=1}^{2} \frac{n^2}{2^n} = \frac{2(2^2)(2-1)^0 + 1(2^1)(2-1)^1}{(2-1)^3} = 6 \]

and

\[ S(4,5) = \sum_{n=1}^{4} \frac{n^4}{5^n} = \frac{24(4^0)(5^4) - 36(4^1)(5^3) + 14(4^2)(5^2) - (4^3)(5^1)}{4^5} = \frac{285}{128} \]