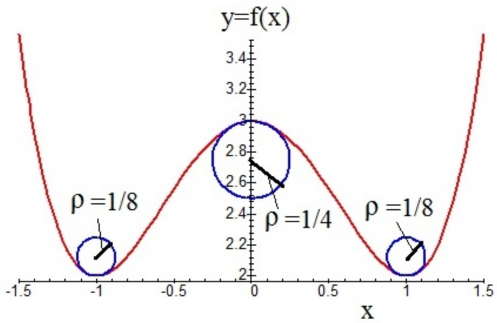
RADIUS OF CURVATURE AND EVOLUTE OF THE FUNCTION y=f(x)

In introductory calculus one learns about the curvature of a function y=f(x) and also about the path (evolute) that the center of curvature traces out as x is varied along the original curve. Beginning students sometimes have difficulty in deriving and understanding the quantity $[1+(df/dx)^2]^{3/2}$which enters such calculations. We want here to review these concepts and add a few simplifying thoughts which will help clarify the problem.

We start with a continuous curve y=f(x) as shown –

FUNCTION $y=x^4-2x^2+3$ SHOWING RADIUS OF CURVATURE AT x=-1, 0, AND +1



An increment of length ds along this curve is-

$$ds=\sqrt{dx^2+dy^2}=dx\sqrt{1+(\frac{df}{dx})^2}$$

Also we have that at x the angle between the x axis and the curve is-

$$\theta=\arctan(\frac{dy}{dx})=\arctan[f'(x)]$$

Differentiating this last expression with respect to x produces-

$$d\theta=\frac{f''(x)}{[1+f'(x)^2]}\mathrm{dx}$$

If we now go back to the above figure, we see that at any point x along the curve y=f(x) there lies a unique circle whose radius represents the radius of curvature r=ρ. Mathematically we have-

$$\rho = \frac{ds}{d\theta} = \frac{dx\sqrt{1+f'(x)^2}}{\{f''(x)dx/[\sqrt{1+f'(x)^2}\,]} = \left| \frac{[1+f'(x)^2]^{3/2}}{f''(x)} \right|$$

The absolute value symbol has been added because ρ should always be considered a positive quantity. Notice this radius of curvature is just the reciprocal of standard curvature, usually, designated by K. The curvature of f(x) changes sign as one passes through an inflection point where f ”(x)=0. In the above example such inflection points occur at x=±1/2. There the radius of curvature ρ becomes infinite and the curvature K=0.

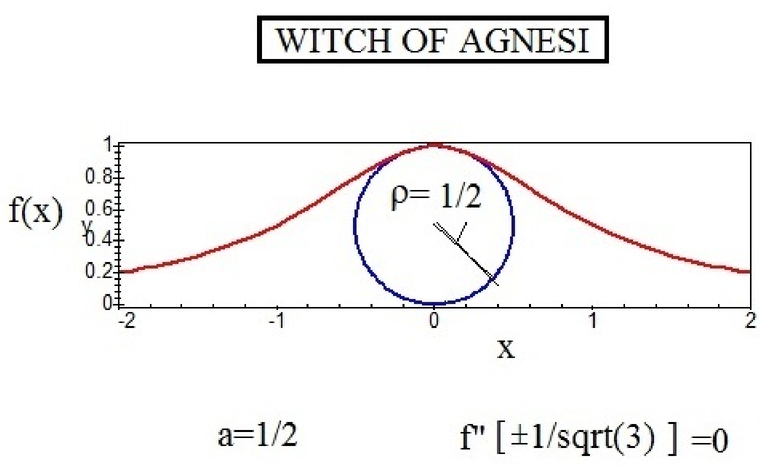
Consider now the following simple curve of historical interest. It is known as the Witch of Agnesi and has the form-

$$y = f(x) = \frac{8a^3}{x^2 + 4a^2}$$

Maria Agnesi (1718-1799) was a famous female Italian mathematician and polymath. She knew six languages by the time she was eleven and was appointed in later life to a professorship at the University of Bologna .The term “witch” was attached to the curve later because of a mistranslation of the word versoria from Latin to English. Differentiating we find-

$$f'(x) = \frac{-16a^3x}{(x^2+4a^2)^2} \quad and \quad f''(x) = \frac{16a^3(-3x^2+4a^2)}{(x^2+4a^2)^3}$$

To simplify things a bit we now set a=1/2. This means y=f(x)=1/(x^2+1). So the function reaches a maximum value of f(0)=1 . The curve has two inflection points at x=±1/sqrt(3). The smallest radius of curvature occurs at x=0 and has the value ρ=1/2. A plot of the Witch of Agnesi curve follows-



The area under the curve equals-

$$A = \int_{x=-\infty}^{\infty} \frac{8a^3}{x^2+4a^2}dx = 4a^2 \int_{w=-\infty}^{+\infty} \frac{dz}{(z^2+1)} = 4a^2\pi \quad where \quad x = 2az$$

So the total area under the curve just equals four times that of the area of the blue circle shown.

An important second curve derivable from y=f(x) is the evolute Y=g(X). This evolute represents the locus of points which represent the moving center of curvature of f(x). Designating a point on the evolute by [X,Y], one has from simple geometry that-

$$X = x - \rho\sin(\theta) \quad and \quad Y = y + \rho\cos(\theta)$$

, where y=f(x) and tan(θ)=f ’(x). Eliminating the angle between these forms for X and Y, we find-

$$[X-x]^2 + [Y-y]^2 = \frac{[1+f'(x)^2]^3}{[f''(x)]^2}$$

Also taking the ratio, we have –

$$\frac{[X-x]}{[Y-y]} = -f'(x)$$

These last two formulas allow us to express both X and Y as functions of x. The formulas for doing this are-

$$X - x = -\frac{f'(x)[1+f'(x)^2]}{f''(x)} \quad and \quad Y - y = \frac{[1+f'(x)^2]}{f''(x)}$$

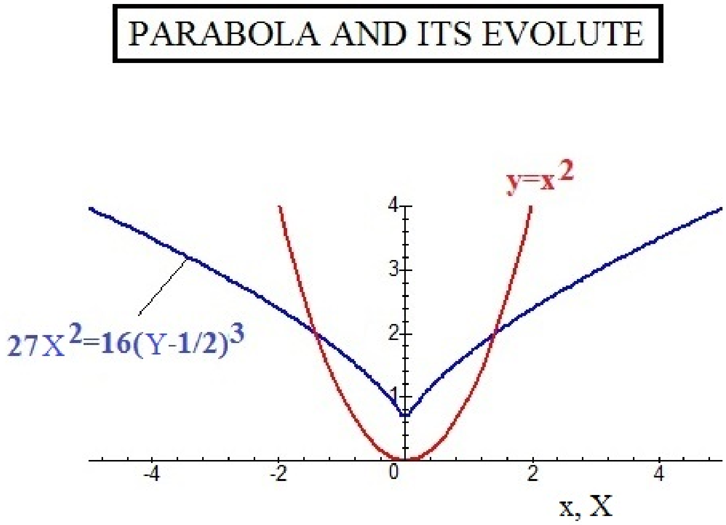
Let us apply this last result to the case of the parabola f(x)=x^2 where f ’(x)=2x and f ”(x)=2. Here we find-

$$X = x - \frac{2x(1+4x^2)}{2} = -4x^3 \quad and \quad Y = x^2 + \frac{1+4x^2}{2} = 3x^2 + (1/2)$$

Eliminating the parameter x from these solutions produces the final result-

$$27X^2 = 16(Y - \frac{1}{2})^3$$

A graph of this evolute and the original parabola follow



Note the cusp at x=0. The presence of such cusps is a common characteristic of evolutes.

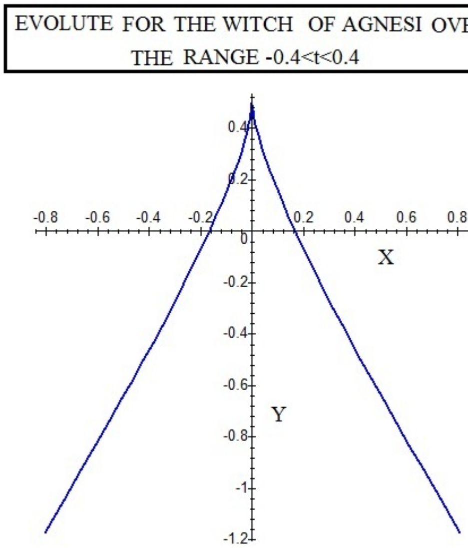
As a second example of an evolute take the Witch of Agnesi curve $f(x)=1/(1+x^2)$. Here the evolute coordinates are given by the rather lengthy parametric form-

$$X=\left(\frac{4t^3}{3}\right)\frac{(3t^2+3t^4+t^6+2)}{(1+t^2)(3t^2-1)} \quad and \quad Y=\left(\frac{1}{2}\right)=\frac{(-1+14t+6t^4+4t^6+t^8)}{(1+t^2)(3t^2-1)}$$

, where we have replaced x by the parametric parameter t. Although one can not eliminate the ts from these two expressions analytically, we can run a standard parametric plot program in MAPLE which reads-

plot([X,Y, t=-0.4..0.4], color=blue, scaling=constrained, thickness=2);

This will produce the following graph instantaneously-



Sixty years ago when I was first introduced to calculus we didn't have PCs and it would have taken many hours to come up with this witches hat or tepee shaped curve.

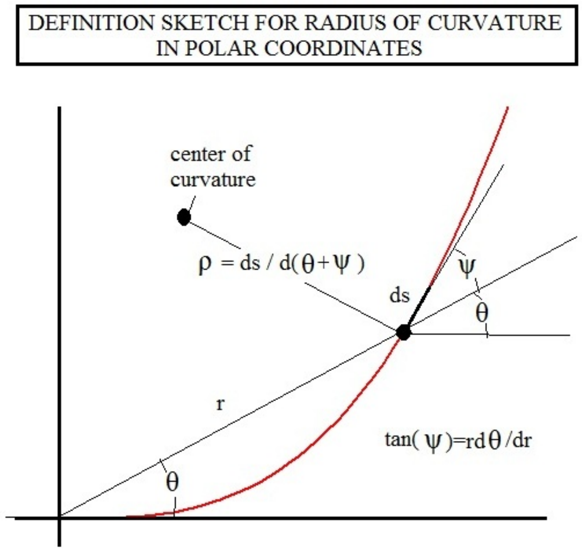
To finish up our discussion on curvature of 2D curves, we look at the radius of curvature of a curve expressed in polar coordinates [r,θ] and related to [x.y] via-

$$x=r\cos(\theta) \quad \text{and} \quad y=r\sin(\theta)$$

This time the increment of length along the curve is-

$$ds = d\theta\sqrt{r^2 + (\frac{dr}{d\theta})^2}$$

To find the angle change one needs , it is first necessary to look at the following sketch-



We see from the sketch that the radius of curvature will be-

$$\rho = \frac{ds}{d(\theta + \psi)} \quad or \quad the \quad equivalent \quad \frac{1}{\rho} = \frac{d\theta}{ds} + \frac{d\psi}{ds}$$

But we have that-

$$d\psi = d\{\cot^{-1}(\frac{dr}{rd\theta})\} = \frac{d\theta}{[r^2 + (\frac{dr}{d\theta})^2]}\left\{r\frac{d^2r}{d\theta^2} - (\frac{dr}{d\theta})^2\right\}$$

After substitutions one ends up with the rather complicated curvature expression-

$$\frac{1}{\rho} = \frac{r^2 + 2(\frac{dr}{d\theta})^2 - r(\frac{d^2r}{d\theta^2})}{[r^2 + (\frac{dr}{d\theta})^2]^{3/2}}$$

This expression usually does not appear in elementary calculus texts because of its complexity. If it does appear it is usually delegated to a homework problem.

For a unit radius circle this result leads to the obvious result that ρ=1. For an Archimedes Spiral, where r=θ , the radius of curvature becomes-

$$\rho = \frac{(1+r^2)^{3/2}}{(2+r^2)}$$

The radius of curvature is thus seen to be directly proportional to the first power of the radial distance from the origin when r gets large.

It is also possible to calculate evolutes of curves expressed in polar coordinates. We will not do so here , but point out the procedure is straight forward and will require arguments similar to those we used above for the Cartesian coordinate cases.