BASIC PROPERTIES OF THE EXPONENTIAL FUNCTION F(x)=Exp(x)

One often encounters the term exponential growth or decay when discussing one of many everyday topics ranging from population growth, to budget deficits , to radioactive waste disposal. The term is often used by the public in a manner indicating an incomplete understanding of the concept. Let us remedy this situation here by giving an elementary and easily understandable discussion of what an exponential function and exponential growth is and how it applies to many different phenomena.

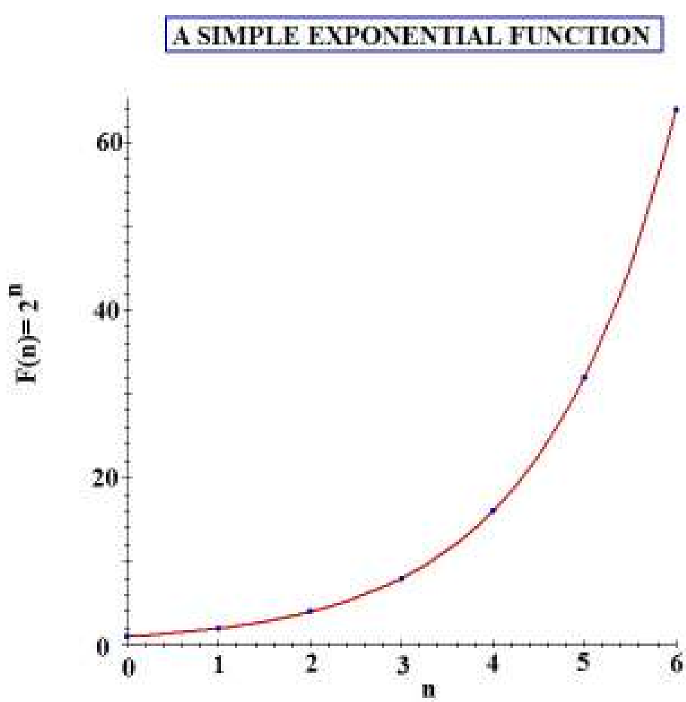
We begin by first looking at the following set of numbers-

$$1 \quad 2 \quad 4 \quad 8 \quad 16 \quad 32 \quad 64 \quad .....$$

It is easy to see that here every number shown equals twice the number directly to its left. Thus the nth number simply equals 2 taken to the nth power. That is, one has the sequence-

$$F(0)=1,\ F(1)=2,\ F(2)=4,\ \textit{and so on with}\ F(n)=2^n$$

A plot of the function F(n) looks as follows-



Here the blue circles represent 2^n for the integer values while the red curve represents the continuous exponential curve valid for all values of n, be they integer or not. As you may recall from your high school or college calculus class, the exponential function $F(x) = 2^x$ will have as its derivative the value-

$$\frac{dF(x)}{dx} = \frac{\lim}{\Delta x \to 0}[\frac{2^{x+\Delta x} - 2^x}{\Delta x}] = 2^x \frac{\lim}{\Delta x \to 0}\left[\frac{e^{\Delta x \ln(2)} - 1}{\Delta x}\right] = 2^x \ln(2)$$

In this expression we have replaced 2^x by $e^{x\ln(2)}$ where e=2.718281828459045.. Here ln refers to the natural logarithm and one has that ln(e)=1. If we replace 2 by e in the above expression, one obtains the simpler result-

$$F(x) = \exp(x) \quad with \quad \frac{dF(x)}{dx} = F(x)$$

That is, the exponential function exp(x) is exactly equal to its derivative. Furthermore, as a corollary, it is also equal to any of its higher derivatives. If one now expands F(x)=exp(x) in a Taylor series about x=0, one finds that-

$$e^x = F(0) + F(0)'\frac{x}{1!} + F(0)''\frac{x^2}{2!}1 + F(0)'''\frac{x^3}{3!} + ...$$
$$= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ... = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Here the primes indicate the order of the derivative. Stopping this expansion after n terms indicates that the error compared to the true value of exp(x) is about $x^{n+1}/(n+1)!$. Thus when I write down from memory that -

$$\exp(1) \approx 2.7182818284590452353602874713526 6$$

I am in fact reproducing an approximation which is obtained by summing things to n=30. Instead of doing this summation via a PC, I simply recall this 32 digit approximation by my own 32 digit mnemonic-

e≈2.7- Adrew Jackson inauguration twice-right triangle-Fibonacci three-full circle-one year before market crash-Boeing's best known jet-end of plague in Europe-route to California. That is-

$$2.7 - 1828{,}1828 - 45{,}90{,}45 - 2{,}3{,}5 - 360 - 28 - 747 - 1352 - 66$$

One can also write e=exp(1) as the following continued fraction-

$$\exp(1) = 1 + 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

This expansion clearly shows that, although e is an irrational number, it contains a simple internal structure. Notice the sequences on the two diagonals of-

$$1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \quad and \quad 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,$$

Let us next look at some areas where exponential growth or decay occurs. Consider first a bacteria colony where it is observed that its population P(t) increases in direct proportion to its size. Mathematically this can be expressed via the first order differential expression-

$$\frac{dP(t)}{dt} = \alpha P(t) \quad subject \quad to \quad P(0) = P_o$$

On integrating this once and applying the initial condition one finds-

$$P(t) = P_o \exp(\alpha t)$$

The bacteria colony thus grows exponentially in time as long as $\alpha>0$. Its path will look similar to the curve for $F(x)=2^x$ shown above. In reality such exponential growth generally will stop after awhile sine the food supply will become insufficient. Similar things can be said for the human population which is now around 7 billion and growing exponentially with a current doubling time of 45 years and hence with an $\alpha=\ln(2)/45$.

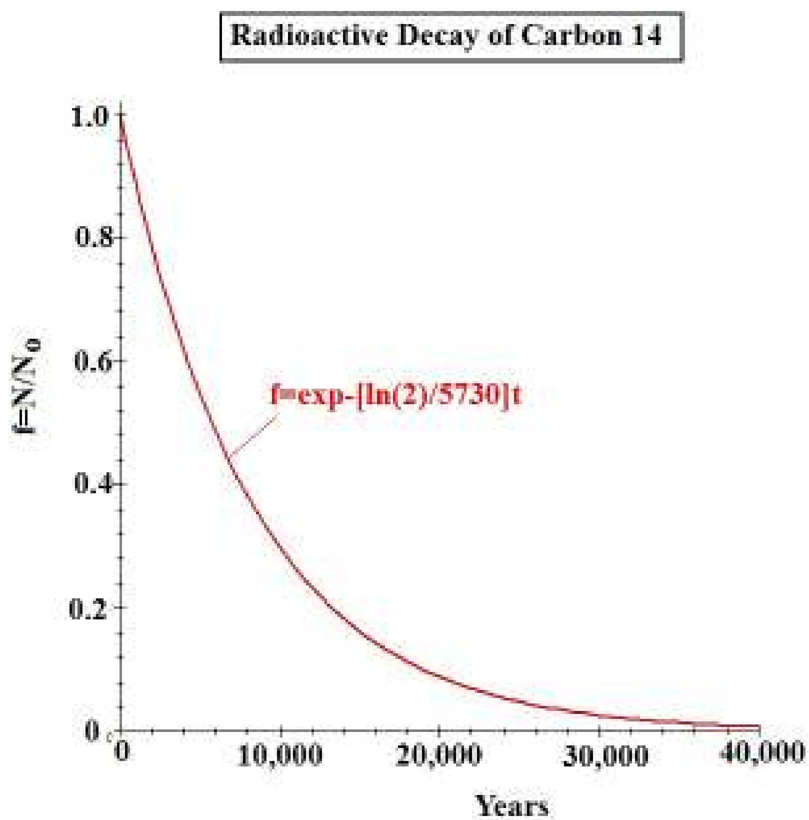
Another problem in which one encounters the exponential function is that of radioactive decay of unstable nuclei. We know, for example, that C14, a radioactive isotope of normal carbon 13, has a half life of about 5730 years. The isotope follows the decay law-

$$-\ln\left[\frac{N}{N_0}\right] = \beta t \text{ or the equivalent } N = N_0 \exp(-\beta t)$$

Here N_o represents the initial amount of C14 present in a specimen, N is the amount left after time t, and β is the decay constant. The decay constant for C14 will be –

$$\beta = \ln(2)/5730 = 1.20968 \cdot 10^{-4} \text{ per year}$$

A plot of the fraction $f=N/N_0$ of C14 remaining after t years is shown in the following graph-



The fraction remaining clearly demonstrates an exponentially decaying process. According to the graph, a 10,000 year old piece of charcoal will have an f= exp(10,000*ln(2)/5730=0.2983.

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This nuclear decay process is quite independent of external environmental factors and therefore may be used to date anything containing carbon provided it was living at one time. This includes human bones, cut wood from trees, and charcoal from fires. The idea behind the process is that every living thing (animal or plant) has a fixed fraction of carbon 14 within their body and this fraction will no longer be replenished after death. As a result dates in the range of 100 years to about 60,000 years ago can be nicely dated. For example the charcoal left behind in some of the cave dwellings in France dates the early cave painters as having lived about 30,000 years before the present. Stonehenge and the Egyptian Pyramids have also been precisely dated via the radioactive carbon method. Of course radioactive decay can also be harmful as occurs during nuclear reactor accidents such as those at Chernobyl and more recently at Fukushima. The spewing of radioactive isotopes of iodine, cesium and plutonium into the atmosphere and ground water can lead to their uptake in the food chain with subsequent health effects such as cancers and birth defects in humans.

Finally I show you a way to quickly estimate the values of F(x)=exp(x) using a little trick I recently discovered. The idea is to look at the following integral-

$$\int_{x=0}^{1} P_{2n}(x)\cosh(ax)dx = [-\exp(-a)N(n,a) + \exp(a)M(n,a)]/a^{2n+1}$$

where $P_{2n}(x)$ is the even 2nth order Legendre Polynomial and N(a,n) and M(a,n) are polynomials involving 'a's for a given value of n. If n is large enough and 'a' small enough the integral multiplied by a^{2n+1} exp(a) will approach zero and so one obtains the approximation-

$$\exp(a) \approx \sqrt{\frac{N(n,a)}{M(n,a)}}$$

Applying this procedure for n=20 and a=1, yields the 46 digit accurate approximation-

$$e \approx \sqrt{\frac{8583196779242037169211141}{1161609367194302920784 11}} =$$
$$2.718281828459045235360287471352662497757247 09...$$

It would require the summing up of the first 39 terms in the infinite series representation for exp(1) to match this accuracy.