THE FUNCTION EXP(z) AND ITS PROPERTIES

One of the earliest things students learn in their introductory calculus course is that the only function which is exactly equal to its derivative is $F(x) = \exp(x)$. The argument for this starts with the definition for the derivative of $f(x) = b^x$ producing the equality-

$$
\frac{d}{dx} (b^x) \equiv [\ln(b)] \cdot b^x = \lim_{\Delta x \to 0} \left[ \frac{b^{x+\Delta x} - b^x}{\Delta x} \right]
$$

or the equivalent-

$$
b = \lim_{\Delta x \to 0} \left[ 1 + \frac{\Delta x}{\Delta x} \right]^{1/\Delta x} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ... = \sum_{n=0}^{\infty} \frac{1}{n!}
$$

using the binomial expansion and letting $\Delta x$ vanish. Evaluating the sum, one comes up with the irrational number-

$$
b = e = \exp(1) = 2.718281828459045....
$$

Replacing $x$ by $z$ one also has-

$$
F(z) = \exp(z) = \sum_{n=0}^{\infty} \frac{(z)^n}{n!}
$$

where $z$ can be complex. Thus for $z = i\pi$ one has well known identity-

$$
\exp(i\pi) = -1
$$

It also follows that-

$$
\exp(iz) = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \cos(z) + i\sin(z)
$$

and-

$$
\exp(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = \cosh(z) + \sinh(z)
$$

A graph of $\exp(x)$ follows-
From the above results one has at once that-

\[\exp(iz) + \exp(-iz) = 2\cos(z) \quad \text{and} \quad \exp(iz) - \exp(-iz) = 2i\sin(z) \quad \text{also} \]

\[\exp(z) + \exp(-z) = 2\cosh(z) \quad \text{and} \quad \exp(z) - \exp(-z) = 2\sinh(z) \]

Note that \(\cos(z)\) and \(\cosh(z)\) have an even symmetry since replacing \(z\) by \(-z\) does not change the definition. The functions \(\sin(z)\) and \(\sinh(z)\) both vanish at \(z=0\) and have the odd symmetry \(\sin(-z)=-\sin(z)\) and \(\sinh(-z)=-\sinh(z)\). Double angle formulas are readily established by noting that-

\[\cos(z) = \Re[\exp(iz)], \quad \text{and} \quad \sin(z) = \Im[\exp(iz)]\]

So, for example,-

\[\cos(A + B) = \Re[\exp(i(A + B))] = \Re[(\cos(A) + i\sin(A))(\cos(B) + i\sin(B))]\]
\[= \cos(A)\cos(B) - \sin(A)\sin(B)\]

and-
\[
\sin(A + B) = \frac{1}{2i} \text{Im}[\exp(i(A + B) - \exp(-i(A + B))] = \frac{-i}{2} \text{Im}[e^{iA}e^{iB} - e^{-iA}e^{-iB}]
\]
\[
= \frac{-i}{2} [2i(\sin A\cos B + \cos A\sin B)] = \sin A\cos B + \sin B\cos A
\]

Next consider constructing a curve within the polar coordinate plane where the radial distance \( r \) from the origin is one at \( \theta = 0 \), \( e \) at \( \theta = \pi/2 \), \( e^2 \) at \( \theta = \pi \) etc. These requirements produce the famous Bernoulli exponential spiral:

\[
r = \exp(2\theta / \pi)
\]

A graph of this spiral from \( \theta = 0 \) out to \( \theta = 2\pi \) follows:

**The Exponential Spiral**

One can slow down the rate of unwinding of this spiral by replacing \( 2/\pi \) with a smaller number. The spiral \( r = \exp[\theta/(8\pi)] \) looks as follows:
One can express the series for $\exp(x)$ as a continued fraction expansion:

$$\exp(x) = 1 + \frac{x}{1 - \frac{x}{2 + \frac{x}{3 - \frac{x}{2 + \frac{x}{5 - \frac{x}{2 + \frac{x}{7 - \frac{x}{2 + \frac{x}{9 - \frac{x}{2 + \ldots}}}}}}}}}}$$

which in longhand reads:

$$\exp(x) = 1 + \frac{x}{1 - \frac{x}{2 + \frac{x}{3 - \frac{x}{2 + \frac{x}{5 - \frac{x}{2 + \frac{x}{7 - \frac{x}{2 + \frac{x}{9 - \frac{x}{2 + \ldots}}}}}}}}}}$$
The pattern is clear with the initiation of each subsequent row after the first going as 1-2-3-2-5-2-7-2-9 and the quotient in x following 2 being + and that following the odd numbers being -. Taking up through the 7th row, one finds the approximation $e=\exp(1)=2.718283582$ which is good to five places.

Sums involving $\exp(az)$ can also be evaluated very often using the geometric series. One has-

$$\sum_{n=0}^{\infty} \left( \frac{1}{\exp(z)} \right)^n = \frac{\exp(z)}{\exp(z) - 1}$$

So that-

$$1 + \frac{1}{e^1} + \frac{1}{e^2} + \frac{1}{e^3} + \ldots = \frac{e}{e-1} = 1.5819767068693264243850020051\ldots$$

Also one finds that-

$$\sum_{n=0}^{\infty} \exp(-(2n+1)) = \frac{1}{2\sinh(1)} \quad \text{and} \quad \sum_{n=1}^{\infty} \left( \frac{1 - 6e^{-2x} + e^{-4x}}{1 + 2e^{-2x} + e^{-4x}} \right)^n = \frac{1}{4}[1 + \cosh(2x)]$$

and-

$$\sum_{n=0}^{\infty} \left( \frac{1 - \exp(-x)}{1 + \exp(-x)} \right)^n = \frac{1 + \exp(x)}{2}$$

Integrals involving $\exp(x)$ can often be handled by means of Laplace Transforms. One has-

$$\int_{t=0}^{\infty} f(t) \exp(-st) dt = \text{Laplace}(f(t))$$

Thus one finds-
\[ \int_{x=0}^{\infty} J_0(x) \exp(-x) \, dx = \frac{1}{\sqrt{2}}, \quad \int_{x=0}^{\infty} x^2 \sin(x) \exp(-x) \, dx = \frac{1}{2} \quad \text{and} \]
\[ \sqrt{\pi} = 2 \exp\left(\frac{1}{4}\right) \int_{x=0}^{\infty} \cos(x) \exp(-x^2) \, dx \]

Other types of integrals involving the exponential function include-

\[ \int \sin(bx) e^{ax} \, dx = \text{Im}\left[\int \exp(a + ib)x \, dx\right] = \text{Im}\left[\frac{\exp(a + ib)x}{a + ib}\right] \]
\[ = \frac{e^{ax}}{a^2 + b^2} [a \sin(bx) - b \cos(bx)] \]

and-

\[ \int \cos(bx) e^{ax} \, dx = \text{Re}\left[\frac{\exp(a + ib)x}{a + ib}\right] = \frac{e^{ax}}{a^2 + b^2} [a \cos(bx) + b \sin(bx)] \]

We thus have-

\[ \int_{t=0}^{\infty} \sin(bt) \exp(-st) \, dt = \text{Laplace}[\sin(bt)] = \frac{b}{b^2 + s^2} \]

and-

\[ \int_{t=0}^{\infty} \cos(bt) \exp(-st) \, dt = \text{Laplace}[\cos(bt)] = \frac{s}{b^2 + s^2} \]

For the actual numerical evaluation of \( \exp(x) \) one can use the standard Taylor series-

\[ \exp(x) = 1 + \frac{x}{1!} + \frac{(x)^2}{2!} + \frac{(x)^3}{3!} + \ldots \]

and telescope two consecutive terms at a time producing the iteration-

\[ S_{n+1} = S_n + \frac{(x)^{2n} (2n + 1 + x)}{(2n + 1)!} \quad \text{with} \quad S_0 = 0 \quad \text{and} \quad S_1 = \frac{(1 + x)}{1!} \]
For \( x=1 \), this produces the following output-

EVALUATING \( e \) VIA THE ITERATION \( S[n+1]=S[n]+2(n+1)/(2n+1)! \) WITH \( S[0]=0 \), \( S[1]=2 \)

(Value terminated at point where \( S[n] \) departs from \( \exp(1) \))

\[
\begin{align*}
S_1 &= 2.71 \\
S_2 &= 2.7182 \\
S_3 &= 2.718281 \\
S_4 &= 2.718281828 \\
S_5 &= 2.71828182845 \\
S_6 &= 2.7182818284559045 \\
S_7 &= 2.71828182845904523 \\
S_8 &= 2.71828182845904523533 \\
S_9 &= 2.71828182845904523536028 \\
S_{10} &= 2.71828182845904523536026 \\
S_{11} &= 2.718281828459045235360274 \\
S_{12} &= 2.7182818284590452353602874 \\
S_{13} &= 2.71828182845904523536028713 \\
S_{14} &= 2.71828182845904523536028713526 \\
S_{15} &= 2.718281828459045235360287135266249 \\
S_{16} &= 2.718281828459045235360287135266249775 \\
S_{17} &= 2.718281828459045235360287135266249775704 \\
S_{18} &= 2.718281828459045235360287135266249775704709 \\
S_{19} &= 2.7182818284590452353602871352662497757047093700 \end{align*}
\]

50 decimal accuracy achieved after \( n=20 \) iteration

Notice the iteration improves the value of \( e \) by about two decimal points per iteration. I have made up a simple mnemonic allowing me to recall the value of \( \exp(1) \) to 32 places of accuracy from memory. It extends existing mnemonics and reads-

2.7- Andrew Jackson inauguration twice-45 degree right triangle-three term
Fibonacci sequence-full circle in degrees-year before the stock market crash-Boeing
famous jet-end of black death in Europe-famous route to California.

That is- \[2.7182818284590452353602871352662497757047093700 \]

It is also possible to continue the telescoping by taking four terms at a time. This produces the series-

\[
\exp(x) = \sum_{n=0}^{\infty} \frac{x^n [64n^3 + 16n^2 (6 + x) + 4n(x^2 + 5x + 11) + x^3 + 3x^2 + 6x + 6]}{(4n + 3)!}
\]
and-

\[ e = \sum_{n=0}^{\infty} \frac{4(16n^3 + 28n^2 + 17n + 4)}{(4n + 3)!} = \frac{8}{3} + \frac{13}{252} + \ldots \approx \frac{685}{252} \]

Although these series converge at an accelerated rate, the extra complication appearing in the numerator term clearly suggests further telescoping will no longer offer additional advantages.

Some further improvement in the convergence rate can be achieved by iterating the series by setting \( x = \frac{1}{k} \) with \( k > 1 \) and then taking the \( k \)th power of the result. In most instances this power taking procedure is not worth the extra calculation time involved. A few extra iterations on \( S_n \) will work just as well. Higher powers of \( e \) follow by multiplication. Thus –

\[ \exp(10) = [\exp(1)]^{10} = e \cdot (e^{2\cdot e^2})^2 = 22026.46579480671651695790\ldots \]

Finally we recall the basic inverse relation between \( \exp(x) \) and its inverse. One has-

\[ N = \exp(x) \quad \text{with} \quad x = \ln(N) \]

So that the logarithm of \( N = 2 \) to the base \( e \) is \( x \), so that \( 2 = \exp(x) \). A simple application of the Newton-Raphson Method then yields-

\[ x_{n+1} = x_n - 1 + 2 \exp(-x_n) \quad \text{with} \quad x_0 = \frac{2}{e} \]

and produces- \( x = 0.693147180559945309417232121458\ldots \) after just five iterations. Likewise for \( 10 = \exp(x) \), the Newton Raphson method, starting with \( x_0 = 1 \), produces the thirty digit accurate result-

\[ \ln(10) = 2.30258509299404568401799145468\ldots \]

after eight iterations. Using this result we also have-

\[ \exp(p) = 10^p / \ln(10) = 10^{0.43429448\ldots} p \]

which means that \( \exp(100) = 10^{43.429448\ldots} = 2.6881172\ldots \cdot 10^{43} \).

September 2009