VARIATIONS ON THE FIBONACCI SEQUENCE

One of the best known mathematical sequences is that of Fibonacci .It reads-

$$S(n)=\{1,2,3,5,8,13,21,33,54,\ldots,f[n]\}$$

The elements f[n] in this sequence are determined by a solution of the difference equation-

$$f[n+2]=f[n+1]+f[n] \quad \text{subject to } f[1]=1 \text{ and } f[2]=2$$

We have –

$$\begin{aligned}
f[3] &= f[1]+f[2]=3\\
f[4] &= f[1]+2f[2]=5\\
f[5] &= 2f[1]+3f[2]=8\\
f[6] &= 3f[1]+5f[2]=13\\
f[7] &= 5f[1]+8f[2]=21\\
f[8] &= 8f[1]+13f[2]=34
\end{aligned}$$

In matrix form this last set of equations may be written as-

$$\begin{bmatrix}1 & 1\\ 1 & 2\\ 2 & 3\\ 3 & 5\\ 5 & 8\\ 8 & 13\end{bmatrix}\cdot\begin{bmatrix}f[1]\\ f[2]\end{bmatrix}=\begin{bmatrix}f[3]\\ f[4]\\ f[5]\\ f[6]\\ f[7]\\ f[8]\end{bmatrix}$$

The solutions to this equation are the Fibonacci Numbers f[3]=3, f[4[=5, f[5]=8, f[6]=13, f[7]=21, f[8]=34. Other sets of f[n] values can be found by replacing the starting conditions $[1\ 2]^T$ by say $[1\ 3]^T$. This change leads to the Lucas Numbers {1,3,4,7,11,18,29,47,…}. Regardless of the starting conditions $[f[1]\ \ f[2]]^T$ the ratio of f[n+1]/f[n] as n goes to infinity equals the golden ratio φ=(1+sqrt(5))/2=1.6180339… .

The purpose of this note is to look at some variations of the Fibonacci Sequence which appear not to have received attention in the literature. One such a modification is governed by the finite difference equation-

$$f[n+4]=f[n+1]+f[n+2]+f[n+3] \quad \text{subject to the starting values } f[1], f[2], \text{ and } f[3] .$$

The solution to this difference equation, good through f[9] , can be found in solving thris given in matrix form -

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 4 \\ 4 & 6 & 7 \\ 7 & 11 & 13 \\ 13 & 20 & 24 \end{bmatrix} \cdot \begin{bmatrix} f[1] \\ f[2] \\ f[3] \end{bmatrix} = \begin{bmatrix} f[4] \\ f[5] \\ f[6] \\ f[7] \\ f[8] \\ f[9] \end{bmatrix}$$

One can read off at once that f[8]=7f[1]+11f[2]+13f[3]. So if we set f[1]=1,f[2]=2 and f[3]=3, one finds f[8]=68. The elements in this 6 by 3 matrix are generated as follows-

a[n,3] =a[n+1,1]   a[n,1]+a[n,3]=a[n+1,2]   a[n,2]+a[n,3]=a[n+1,3]

So the elements in row n=5 become 7, 4+7=11 and 6+7=13. The elements in the 7^{th} row would be [24,37,44] and in the 8^{th} row would be [44,68,81]. For the specific starting conditions f[1]=1,f[2]=2 and f[3]=3, the solution sequence reads-

S={1,2,3,6,11,20,37,68,125}

By increasing the number of rows in the left matrix above one gets the longer sequence-

S={1,2,3,6,11,20,37,68,125,230,423,778,1431,2632,4841,8904,16377,30122,…}

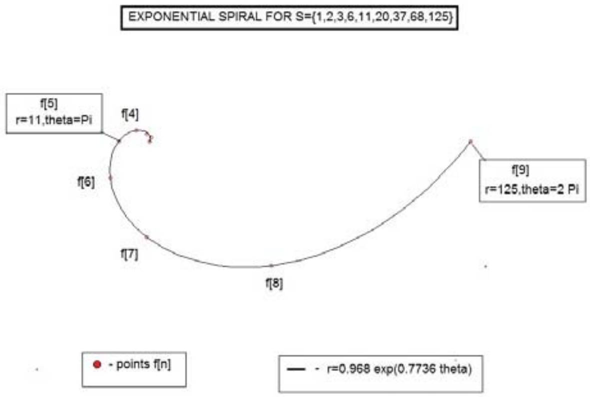
The ratio R=f[n+1]/f[n] of neighboring fs is given by looking at the ratio of neighboring elements in S. For the largest elements shown above we get R=30122/16377=1.8392868. Using my PC the calculzate f[200] anf f[199] we get an improved ratio–

R=1.83928675521416113255185256465328660042417874609759224677875 8…

Note that this value is independent of the three starting values {f[1],f[2],f[3]}. Here R plays the same role as does the golden ratio for the standard Fibonacci Sequence. Monotonically increasing sequences such as S above can be represented graphically as exponential spirals when neighboring elements are separated by a specified angle from each other. For S={1,2,3,6.11,20,37,68,125} we can use the exponential representation –

r=a exp(-bθ)   with b=(1/π)ln(125/11)] and a=11exp(-bπ)

provided that the angle separation between f[n+] and f[n}=π/4. Here is the corresponding graph-



As another variation on the Fibonacci Sequence, consider the difference equation-

f[n+5]==f{n+1]+f[n+2]+_f[n+3]+f[n+4] subject to f[1]=1, f[2]=2, f[3]=3 and f[4]=4

Here the first few terms in the corresponding sequence are –

S={1,2,3,4,10, 19, 36,69,134,258,437,…}

The corresponding coefficient matrix reads-

$$M=\begin{bmatrix}1&1&1&1\\1&2&2&2\\2&3&4&4\\4&6&7&8\\8&12&14&15\\15&23&27&29\end{bmatrix}$$

The elements a[n,m] in this matrix are easy to find. We have-

a[n,4]=a[n+1,1] a[n,1]+a[n,4]=a[n+1,2] a[n,2]+a[n,4]=a[n+1,3] a[n,3]+a[n,4]=a[n+1,4]

So that the next row in the above matrix reads [29,44,52,56]. It says that-

f[11]=29(1)+44(2)+52(3)+56(4)=497

The ratio of f[n+1]/f[n] as n gets large yields-

R= :=1.92756197548292530426190586173662216869855425516338472714664703…

One suspects that this ratio will reach R=2 for a difference equation of the form-

$$f[n+m]=\sum_{k=1}^{m-1} f[n+k] \quad \text{as} \quad m \text{ goes to infinity}$$

This observation follows from the fact that 1.618<1.839<1.927.

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