GENERALIZATION OF THE FIBONACCI SEQUENCE

One of the best known mathematical sequences is the Fibonacci sequence-
1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610,...

where subsequent terms are generated by the formula-

\[ f_{n+2} = f_{n+1} + f_n \quad \text{with} \quad f_1 = 1 \text{ and } f_2 = 2 \]

Fibonacci (alias Leonardo of Pisa) introduced this sequence in his 1202 mathematics book "Liber Abaci" which also introduced the European world to the concept of zero. As first noticed by Johannes Kepler (of astronomy fame), the ratio of \( f_{n+1}/f_n \) as \( n \) goes to infinity equals the Golden Ratio. That is,

\[
\lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \frac{1 + \sqrt{5}}{2} = 1.6180339887...
\]

Also one notes the following -

\[
\sum_{k=1}^{3} f_k = 6 = f_5 - 2
\]
\[
\sum_{k=1}^{4} f_k = 11 = f_6 - 2
\]
\[
\sum_{k=1}^{5} f_k = 19 = f_7 - 2
\]
\[
\sum_{k=1}^{6} f_k = 32 = f_8 - 2
\]

From these identities one may conclude that –

\[
\sum_{k=1}^{N} f_k = f_{N+2} - 2
\]

Thus-

\[1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 + 89 + 144 + 233 = 610 - 2 = 608\]
Also one notes that the Fibonacci Numbers $f_{2+3k}$ are all even for $k=0,1,2,3,\ldots$ . A little manipulation then shows that-

$$f_{5+3k} = 3f_{2+3k} + 2f_{1+3k}$$

Since $f_{23}=46368$, $f_{20}=10946$, and $f_{19}=6765$, one has-

$$46368 = 3(10946) + 2(6765)$$

There are many more identities involving Fibonacci sequences and numbers. The reader is referred to http://en.wikipedia.org/wiki/Fibonacci_number for some of these. (Note that their definition for the $f$s differs slightly from that used here)

Let us now get to the main topic of this page, namely, to present a variation on the Fibonacci sequence. We consider a sequence were the first $N$ numbers are given by $g_m=1$, 2, 3,…,N and the $g_{N+1}$ term is defined as their sum. That is-

$$g_{N+1} = \sum_{m=1}^{N} g_m = 1 + 2 + 3 + \ldots + N = \frac{N(N+1)}{2}$$

Following the usual Fibonacci procedure the next two terms become-

$$g_{N+2} = g_{N+1} + \frac{N(N+1)}{2} - g_1 = N^2 + N - 1$$

$$g_{N+3} = g_{N+2} + g_{N+1} + \frac{N(N+1)}{2} - g_1 - g_2 = 2N^2 + 2N - 4$$

We call these gs the NTuple Fibonacci Numbers. For $N=2$ one recovers the classical Fibonacci values $g_3=3$, $g_4=5$, and $g_5=8$. Note that after one reaches $g_{2N}$ the $g_{2N+1}$ term and higher will just be-

$$g_{2N+1} = \sum_{m=N}^{2N} g_m , \quad g_{2N+2} = \sum_{m=N+1}^{2N+1} g_m \quad \text{etc}.$$.

Here are a few of the Generalized Fibonacci Sequences-

for $N = 3$ get $1, 2, 3, 6, 11, 20, 37, 68, 125, 230, 423, 778,$...

$N = 4$ get $1, 2, 3, 4, 10, 19, 36, 69, 134, 258, 497, 958,$...

$N = 5$ get $1, 2, 3, 4, 5, 15, 29, 56, 109, 214, 423, 831,$...
We can plot these on a semi-log graph for $n$ from 1 to 25 to get the three curves shown-

You will notice that as $n$ gets large the curves can be well approximated by the formula-

$$\ln(g_n) = an + b$$

where the constants $a$ and $b$ are found by evaluating this approximation at say $n=30$ and at $n=50$ for a given $N$. We find that for the standard Fibonacci Numbers ($N=2$) that-

$$a = 0.48121 \text{ and } b = -0.3234$$

This estimate yields the approximation at $N=2$ of-

$$g_{100} \approx \exp[0.48121 \cdot 100 - 0.3234] = 5.73104644 \times 10^{20}$$

compared to the exact value-

$$g_{100} = 57314784401381708410$$
This approximation is seen to be accurate to four decimal places. The values for a and b will change with N with the slope (and hence a) of the semi-log increases with increasing value N.

The ratio \( R_N = \frac{g_{n+1}}{g_n} \) for each of these sequences as n gets large goes to a finite number ranging from the Golden ratio for \( N = 1 \) to 2 for \( N \) becoming infinite. The ratio of 2 for large \( N \) follows directly from above by looking at \( g_{N+2}/g_{N+1} \). The ratios for other \( N \) are simplest to obtain numerically by just looking at something like the \( g_{51}/g_{50} \) and comparing with \( g_{50}/g_{49} \). Here are the ratios for \( N = 2, 3, 4, 5, \) and \( \infty \) -

\[
R_2 = 1.61803398874989484\ldots
\]
\[
R_3 = 1.83928675521416113\ldots
\]
\[
R_4 = 1.92756197548292530\ldots
\]
\[
R_5 = 1.96594823664548533\ldots
\]
\[
R_\infty = 2
\]

In looking at the sequence for \( N = 3 \), one notices that all elements \( g_{2n} \) are even and elements \( g_{2n+1} \) are odd. Thus-

\[
g_{48} = 2620397211992 \text{ is even and } g_{49} = 4819661885417 \text{ is odd}
\]

Also in this particular case \( f_{49} \) is a prime number. We also find for \( N = 3 \) that the following identity holds-

\[
g_{8+k} = 24g_{2+k} + 20g_{1+k} + 13g_k \text{ for } k = 1, 2, 3, \ldots
\]

so that-

\[
30122 = 24(778) + 20(423) + 13(230)
\]

Also one finds for \( N = 3 \) that-

\[
\sum_{k=1}^\infty \frac{1}{f_k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{11} + \frac{1}{20} + \ldots = 2.200170\ldots
\]
Notice that if \( N = 2 \), and thus involves the standard Fibonacci Numbers, the sum of the reciprocals will differ from the above value for the \( N = 3 \). Indeed we have taking the reciprocal series for Fibonacci Numbers out to the term \( 1/f_{40} \) that-

\[
\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \frac{1}{13} + \ldots = 2.3598856..
\]

which is a slightly larger sum. One can see that this must be so by subtracting term by term in the two series. We get-

\[0 + 0 + 0 + (1/5 - 1/6) + (1/8 - 1/11) + (1/13 - 1/20) + (1/21 - 1/37) > 0\]

Many additional identities may be developed for \( N = 3 \) and higher. One of these is-

\[g_{10+k} = 29g_{4+k} + 27g_{3+k} + 23g_{2+k} + 15g_{1+k} \quad \text{for} \quad k = 0, 1, 2, 3, \ldots\]

and is valid for \( N = 4 \). Thus-

\[958 = 29(19) + 27(10) + 23(4) + 15(3)\]

And=

\[g_{18} = 29(958) + 27(497) + 23(258) + 15(134) = 49145\]

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