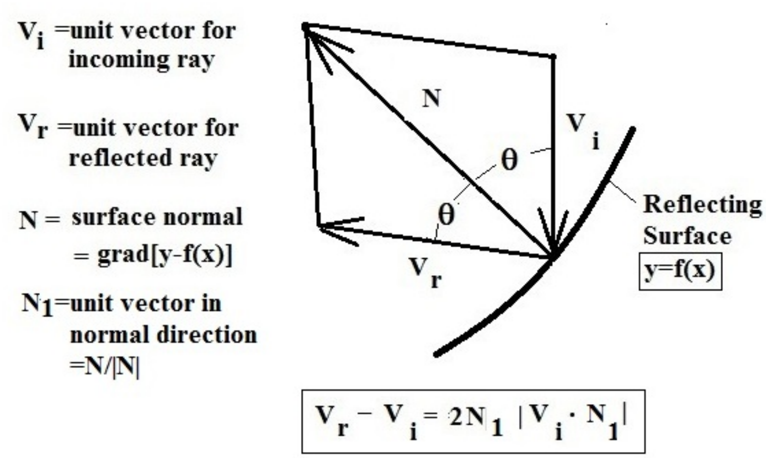
# PARABOLIC AND ELLIPTIC REFLECTORS

It is well known that reflecting surfaces of either parabolic or elliptic shape have the interesting property that incoming light rays with specified orientation can reflect off of such surfaces to produce a bundle of reflected rays which converge at one point termed the focus. We wish here to review the procedure for locating such focal points.

We start the analysis by looking briefly at the basic reflection law from a surface. Since paraboloids and ellipsoids can in most instances be taken as being axisymmetric, it is sufficient to look at just the 2D version of reflection. This is done by first defining a curve y=f(x) as shown in the accompanying figure-

**VECTOR FORM OF THE REFLECTION LAW**




Next we consider an incoming light ray parallel to the y axis. Its unit length vector form equals $V_i$= -j. When it hits the reflecting surface a reflected ray will be established. Its unit length form is denoted by Vr. A third vector N, not of unit length, is formed by taking the gradient to the surface at the impingement point. It equals N=grad[y-f(x)]. Now from optics we know that the reflection angle θ equals the angle of incidence θ. This means the dot products $|N/|N|\cdot V_i|$ and $|N/|N|\cdot V_r|$ both equal to cos(θ). Applying vector addition we then have-

$$V_r + (-V_i) = 2|V_i \cdot N_1|\, N_1$$

, where $N_1$=N/|N| is the unit length vector along N. This equation represents essentially the Law of Reflection for any incoming light ray and can thus be used for both parabolic and elliptic reflectors.

PARABOLIC REFLECTORS:

Here we have a reflector whose surface contains the parabola $y=x^2$. Assume an incoming light ray whose unit length vector reads $V_i$= -j. The Inward normal to this parabola is-

$$N=\mathrm{grad}(y-x2)=-2ix+j$$

and the unit length vector parallel to N will be –

$$N_1 = \frac{-2ix + j}{\sqrt{1+4x^2}}$$

Applying the Reflection Law we find-

$$V_r = -i\left[\frac{4x}{1+4x^2}\right] + j\left[\frac{1-4x^2}{1+4x^2}\right]$$

But we can also write the full reflection vector as-

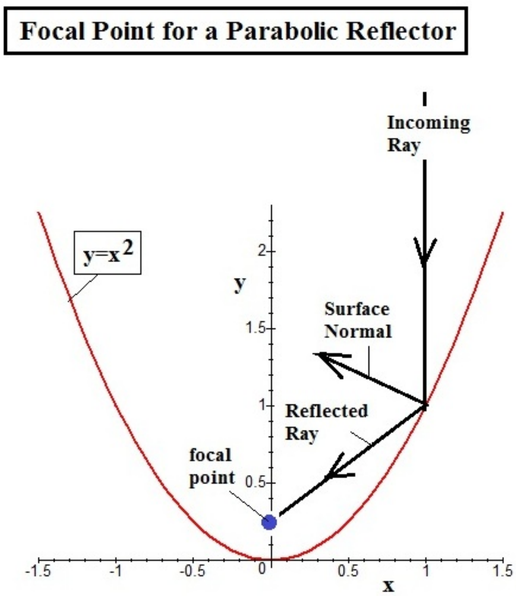
$$(0-x)i+(h-x^2)j=\text{Const. } V_r$$

, where the vector also contains the axial point [x,y]=[0,h].

So we have-

$$\text{Const.}=(1+4x^2)/4 \quad \text{and} \quad h=1/4$$

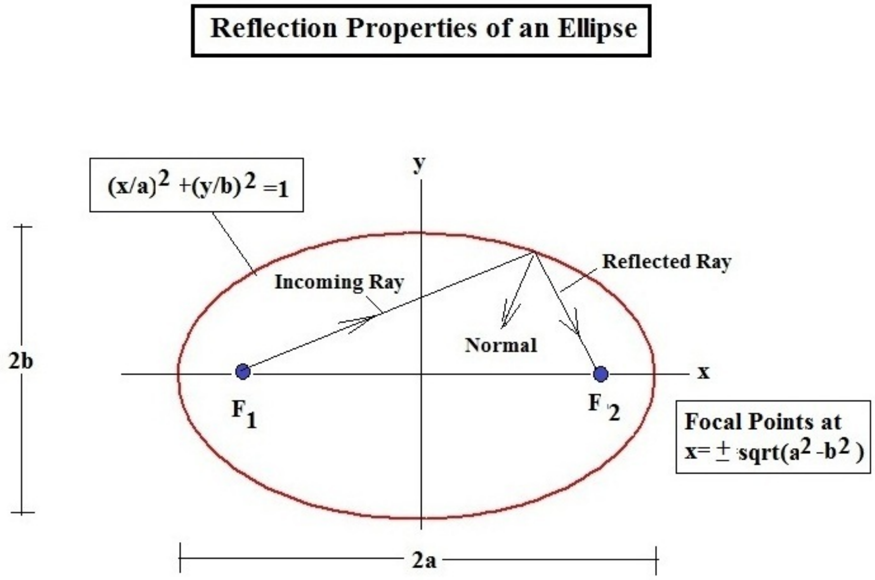
This last result implies that all reflected rays produced by all light rays coming in parallel to the y axis will focus at point x=0 and y=1/4. This is the focal point as also shown in the following figure-




This focusing capability has received wide application in the design of flashlights, auto headlights, acoustic listening devices, and solar trough concentrators. Note that a light ray coming in parallel to the y axis and striking the reflector at [x.y]=[1/2,1/4] will produce a horizontal reflected ray which also passes through the same focal point at [0,1/4]. Large focal length parabolic mirrors form the basis for all reflector telescopes. Such telescopes, like the Hubble space telescope, usually do not place their observation point at the focal point but rather use a secondary reflecting mirror to concentrate the converging light rays at a point behind the primary mirror. Some thirty years ago we were involved with creating some eight foot diameter parabolic mirrors under contract with NASA. We constructed them by first manufacturing a foam composite which approximates a paraboloid and then rotated this form on a turn table after adding a small amount of liquid epoxy to the paraboloid surface. Spinning the form at constant rotational speed until the epoxy hardened led to a perfectly formed parabolic surface which was then coated with reflective mylar. We thus were able to produce near perfect long focal length parabolic mirrors of large diameter at minimal cost. Although these mirrors where not of optical quality, they were perfectly sufficient to concentrate parallel rays of sunlight coming from a heliostat to the order of 300 suns. This concentration was enough to set on fire wooden 2x4s held at a focal point some 30 ft in front of the mirror. We were using the concentrated sunlight to power a Yterium(YAG) glass laser.

ELLIPTIC REFLECTORS:

While parabolic reflectors concentrate parallel light rays to a single focal point, elliptic mirrors have two focal points as shown in the following graph-




The ellipsoid is cut by the ellipse-

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

, whose inward unit  length normal is given by –

$$N_1=\left\{\frac{-\dfrac{ix}{a^2}-\dfrac{jy}{b^2}}{\sqrt{\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}}}\right\}$$

There are two focal points $F_1$ and $F_2$ located at [x,y]=[±sqrt($a^2$-$b^2$),0]. The eccentricity of the ellipse is given by-

$$\varepsilon=\frac{\sqrt{a^2-b^2}}{a}=\frac{c}{a}$$

, where ±c represents the distance from the ellipse origin to its focal points.

Consider now a light ray coming from focal point $F_1$ at [-c,0], hitting the ellipse at [x,y], and then sending a reflected ray toward the x axis. The incoming ray has the unit vector representation –

$$V_i = \frac{(x+c)i + (y-0)j}{\sqrt{(x+c)^2 + y^2}}$$

Using the above stated Law of Reflection allows us to predict that the unit length reflected vector will then have the rather lengthy form-

$$V_r \frac{1}{\sqrt{(x+c)^2+y^2}} \left\{ \begin{array}{l} \left[ (x+c) - \dfrac{2(\frac{x}{a^2})[\frac{x(x+c)}{a^2} + \frac{y^2}{b^2}]}{[\frac{x^2}{a^4} + \frac{y^2}{b^4}]} \right] i + \\ \\ \left[ y - \dfrac{2(\frac{y}{b^2})[\frac{x(x+c)}{a^2} + \frac{y^2}{b^2}]}{[\frac{x^2}{a^4} + \frac{y^2}{b^4}]} \right] j \end{array} \right\}$$

When looking at the special case of [x,y]=[0,b], this produces-

$$V_r = \frac{1}{\sqrt{c^2+b^2}}\{ci - bj\} = (\frac{c}{a})i - (\frac{b}{a})j$$

We can also write the unit vector going from [0,b] to [x*,0] as-

$$V_r = \frac{x^* i - bj}{\sqrt{(x^*)^2 + b^2}}$$

Comparing these two forms for Vr, we can conclude that x*=c. That is, the reflected ray at [0,b] hits the x axis at the second focal point $F_2$ located at [c,0]. This result continues to hold for other reflection points along the ellipse for rays coming from focal point $F_1$. Another, simple to calculate, path is one where the incident light ray comes from $F_1$, impinges at x=c and $y=b^2/a$, and then has a reflected ray $V_r=-j$ passing directly through $F_2$.

Not only does a light source at $F_1$ always hits $F_2$ after reflection but also the transit time is the same and equal to 2a divided by the propagation speed. This means, among other things, that an explosion initiated at $F_1$ will concentrate its energy at the second focus $F_2$. This is essentially the secret behind triggering a hydrogen bomb by ignition of an atomic bomb at $F_1$. It is also the principle behind the very successful and non-invasive pulverizing of kidney stones. Several decades ago one of my students (Mohammad Nasr) studied the focusing capability of a semi-elliptical reflectors immersed in water. By igniting an electric spark at $F_1$ and filming the event with a high speed camera, he observed the rapid growth and then decay of a cavitation bubble at $F_2$ a few hundred microseconds after the spark initiation at $F_{1.}$.

Whispering galleries are found all over the world. They are essentially rooms with ellipsoidal ceilings and circular walls. St Paul's Cathedral in London and the Statuary Hall in the US Capitol are examples of whispering galleries. Two people speaking near focus $F_1$ in one of these rooms can be clearly heard by someone standing a long distance away near $F_2$.

There are other types of reflecting surfaces besides paraboloids and ellipsoids which can cause light or sound waves to both converge and diverge. For example, many years ago we investigated a new type of bell shaped axisymmetric mirror in which a cylindrical radiating light source placed along the mirror axis was capable of producing a high intensity parallel light beam. Such mirrors would be helpful for dentists and physicians examining patients.

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