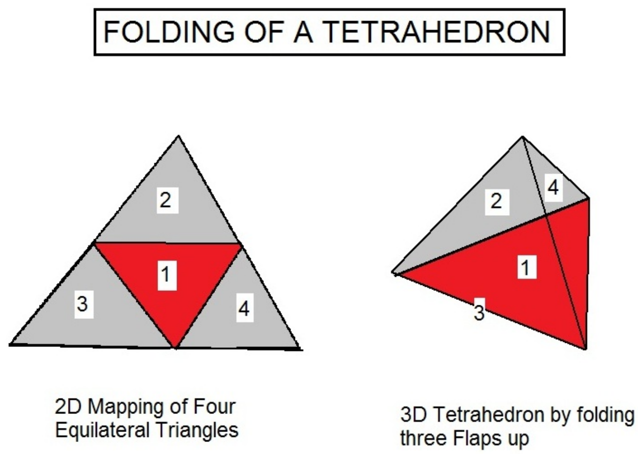
ON THE FOLDING OF 2D PATTERNS

Recently while looking at the construction of 3D polyhedra it became obvious that all Platonic solids can be constructed from 2D patterns. We want here to study this fact in more detail and extend the discussion to other 3D solids with flat faces.

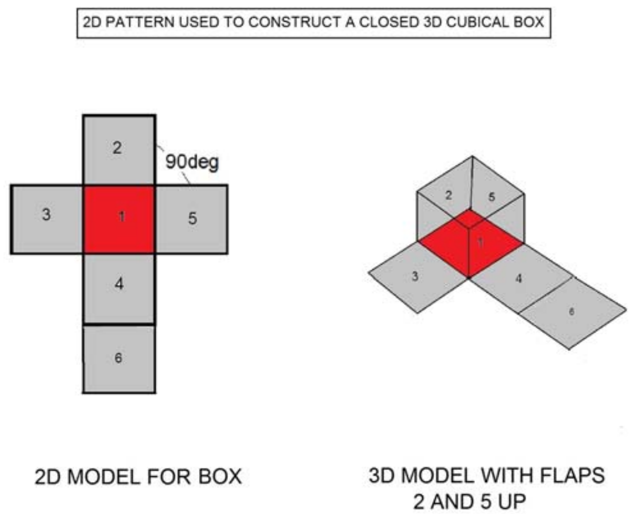
Let us begin with one of the simplest 2D patterns which upon folding leads to 3D tetrahedron structure. It is composed of three equilateral triangles as shown-



When the three external triangles are folded up and made to meet at a common vertex one obtains the tetrahedron shown. This 3D structure consists of a total of four equal triangular faces (F=4). It has a total of four vertices (V=4) and six edges (E=6). It follows the Euler Law valid for any polyhedron that-

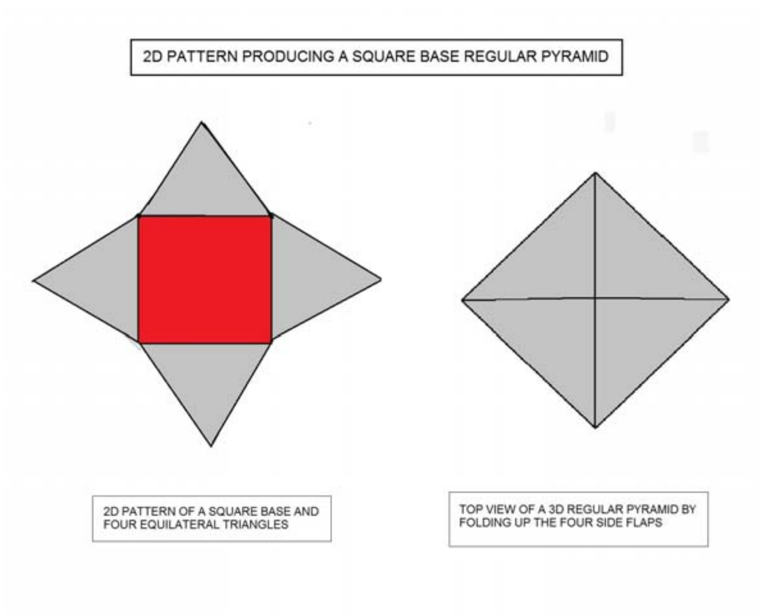
$$V+F-E=2$$

The next pattern we examine is one composed of six squares in form a standard cross as shown-



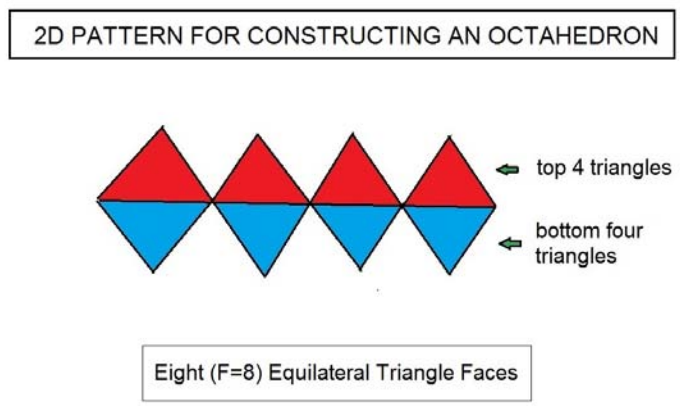
When sides 2 and 5 are folded up we get the 3D structure indicated. Folding up the remaining sides and closing the top flap then produces a 3D cube. This cube has F=6, V=8 and E=12, again consistent with Euler's Law.

We next look at the star like 2D pattern consisting of four equilateral triangles surrounding a square. Here is a picture-



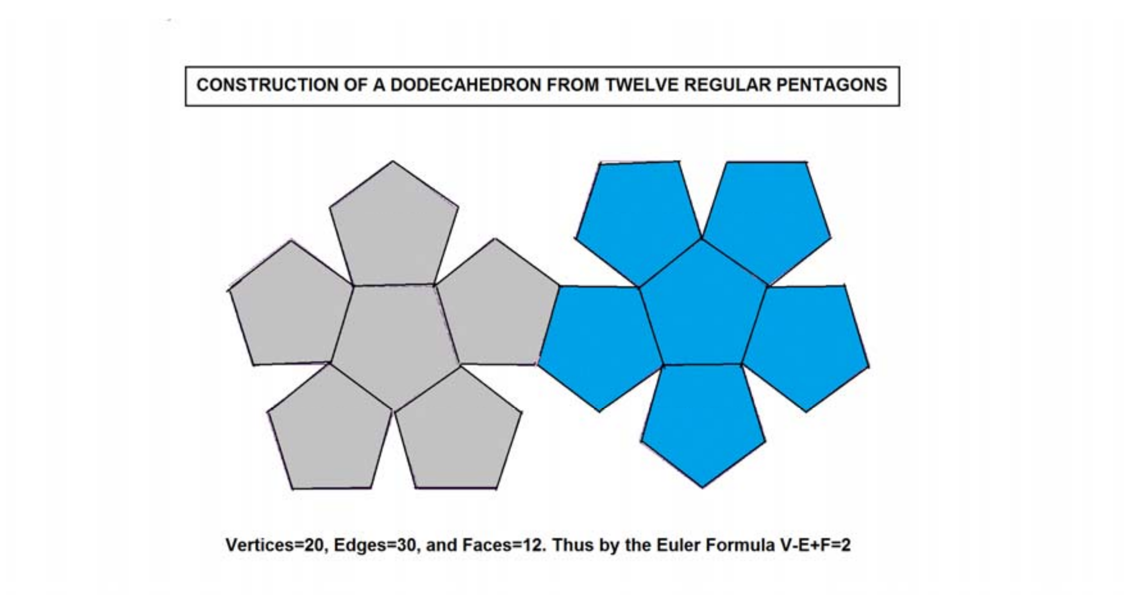
When the four outer flaps are folded up to meet at a common vertex, a standard square base pyramid is formed. In the picture above we show a top view of this pyramid. Note that the resultant solid has five faces(F=5). If the base area is taken as 2 x 2 units, then the pyramid height is exactly H= sqrt(3). The corresponding volume is (2x2)sqrt(3)/3=4/sqrt(3). There are a total of 5 vertices (V=5) and eight edges(E=8).

To construct an octahedron which has a total of eight equilateral triangle faces (F=8), six vertices (V=6), and twelve edges (E=12), we can start with the following 2D pattern-



Upon folding we notice that the open vertex points for the four upper triangles merge into one just as is the case for the bottom four open vertices. So on completion of the folding there will just six unique vertices for this octahedron.

The next Platonic polyhedron where all faces are the same is the dodecahedron consisting of a total of twelve pentagon faces (F=12), with ten vertices(V=10, and twenty edges(E=20). Its construction may be accomplished using the following 2D pattern-

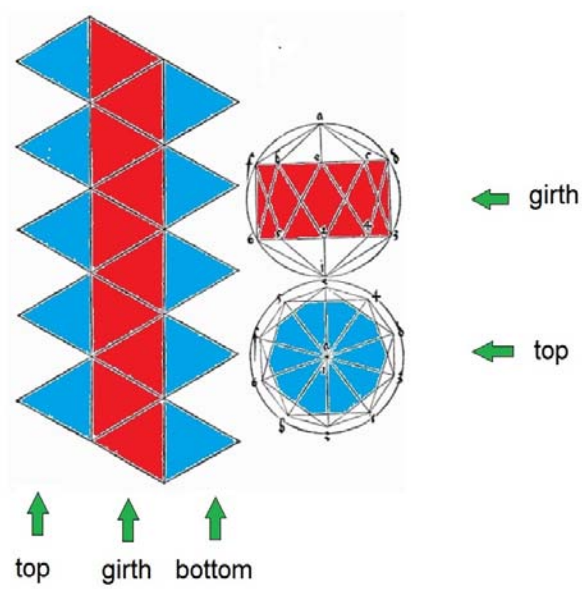


Notice that the external V shaped gaps come together after folding leaving one with a dodecahedron. The ten V shaped gaps in the flat 2D pattern have angle-

$$\varphi=360-3(108)=36deg=(\pi/5)radians$$

The final Platonic Solid is the Icosahedron consisting of twenty equilateral triangle faces (F=20) with twelve vertexes (V=12), and thirty edges (E=30). Its 2D form was first realized by the brilliant German artist Albrecht Durer in his 1525 book on perspective, Here is a slightly modified form of the 2D pattern taken from his book-

ALBRECHT DURER'S 1525 NET FOR CONSTRUCTING AN ICOSAHEDRON

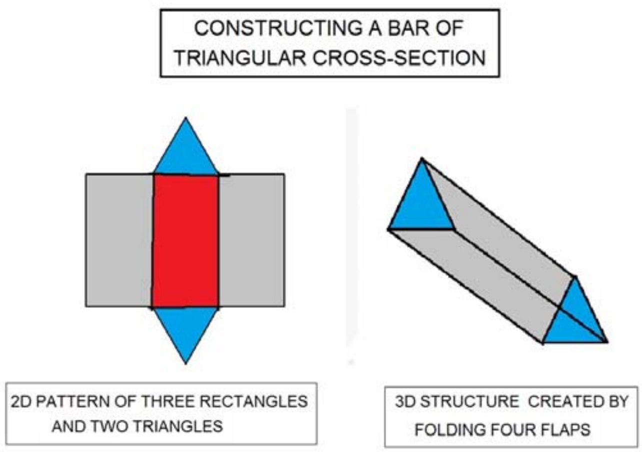


He was perhaps the first artist to recognize how 2D patterns can result in complicated 3D structures by a folding process. There is direct evidence of this in his famous etching Melancholia of 1514. He also exhibited additional mathematical skills by constructing magic squares such as –

13	3	2	16
8	10	11	5
12	6	7	9
1	15	14	4

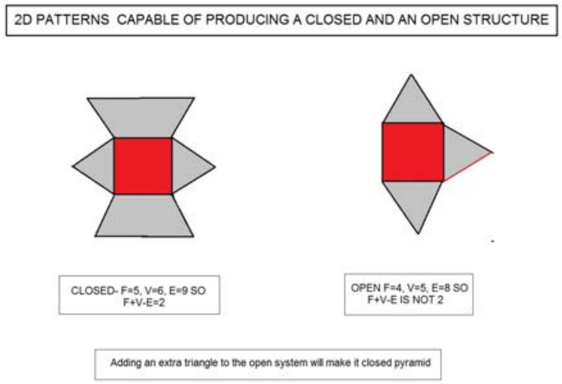
In this square the numbers in any row, column, or diagonal add up to 34. If you look at elements two and three of the fourth row it gives the date of his Melancholia etching. Also if you replace the sum 34 by its reverse 43 you get Durer's age at the time of the etching.

Of all the cases discussed above (with exception of the pyramid) the faces of the final 3D structure where identical. This need not be. It is relatively easy to construct 2D patterns consisting of several different elements which may still be folded into closed 3D structures. One of the simplest of such patterns consists of three equal rectangles plus two isosceles triangles. Here is the 2D pattern and its 3D construct-



We have F=5, V=6, and E=9. This shows that Euler's law is also valid for such unequal surface area 3D constructs.

The remaining questions which concerns us is under what conditions does a 2D pattern guarantee that a completely closed 3D structure results upon folding. To produce such a closed structure clearly requires that neighboring elements to be folded into a common edge have the same edge length and that the height of the outer flaps be large enough to be able to meet upon a single vertex above the base. Also the Euler Law must be satisfied for closed structures. We distinguish between the two cases via the following-



We see here that the first of the patterns leads to a closed structure but the second will be open. To make the second closed will require a fourth triangle in the 2D pattern.

The folding of 2D patterns has been put into an art form by the Japanese. It is referred to as Origami.

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