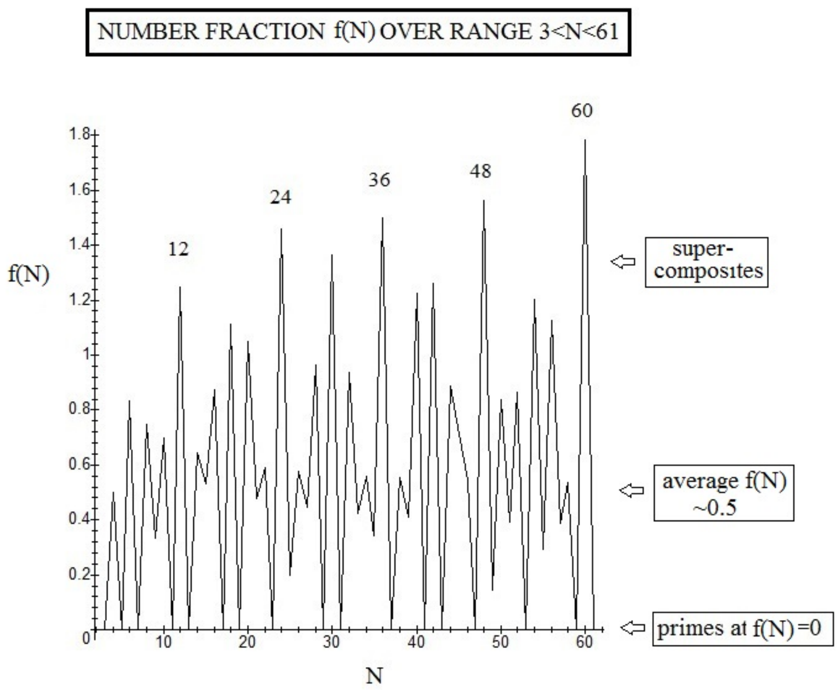
FURTHER PROPERTIES OF NUMBER FRACTIONS

Some four years ago(see- http://www2.mae.ufl.edu/~uhk/NUMBER-FRACTION.pdf) we came up with a new point function termed the number fraction. It is defined as-

$$f(N) = \frac{\sigma(N) - N - 1}{N}$$

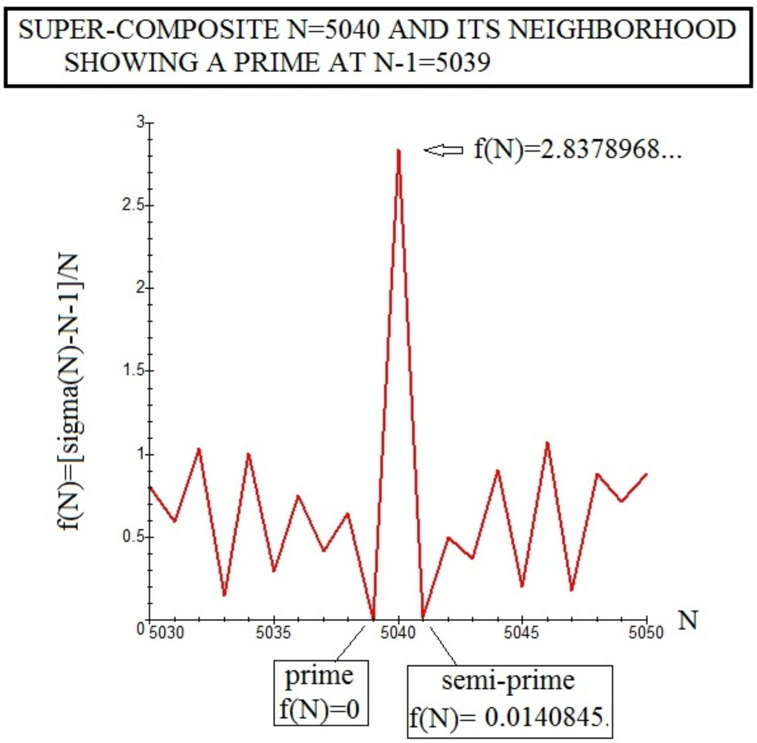
Here σ(N) is the sigma function of number theory which equals the sum of all divisors of a number N. The advantage of the f(N) function over σ(N) is that it will always vanish when N is a prime and have relatively small values even for very large N. It is our purpose here to re-examine this number fraction and thereby determine some of its additional properties.

We begin with the following point plot of f(N) over the range 3<N<61 -



One sees that f(N) vanishes at all primes p=N={3,5,7,11,13,19,23,29,31,37,41,43,47,53,59}within the chosen range. Also there are some local maxima in the f(N) function occurring at multiples of $12=2^2$x3. The average value of the number fraction lies near f(N)=0.5. We term the numbers with f(N) of 1.2 or higher super-composites. This means, for example, that $N=24=2^3$x3 and $60=2^2$x3x5 are super-composites. There are an infinite number of such super-composites just like there are an infinite number of primes. Another interesting observation is that one often finds a prime number removed by just one unit from a super-composite. This occurs for N=12 where 11 and 13 are primes and for N=36 where 37 is a prime.

Looking at some larger numbers N we find that $N=5040=2^4x3^2x5x7$ is a definite super-composite with the unique number fraction f(5040)= 2.8378968253968…A graph of this N in its immediate neighborhood looks as follows-



One observes that f(N) for this super composite stands head and shoulder above its immediate neighbors. Also one find a prime at N-1=5039 plus a semi-prime with a finite but very small value at N=5041. Usually when f(N)<<1 but is not zero then f(N)consists of the product of just two primes. Such numbers are referred to as semi-primes. Another thing to notice is that the average value of f(N) remains near 0.5. Also there appears to be a nearly symmetric behavior in f(N) near a super-composite.

To find other super-composites we note the following properties for those already found-

N	Prime Product	f(N)
60	2^2x3x5	1.783333333…
360	2^3x3^2x5	2.247222222…
840	$2^3x3x5x7$	2.427380952…

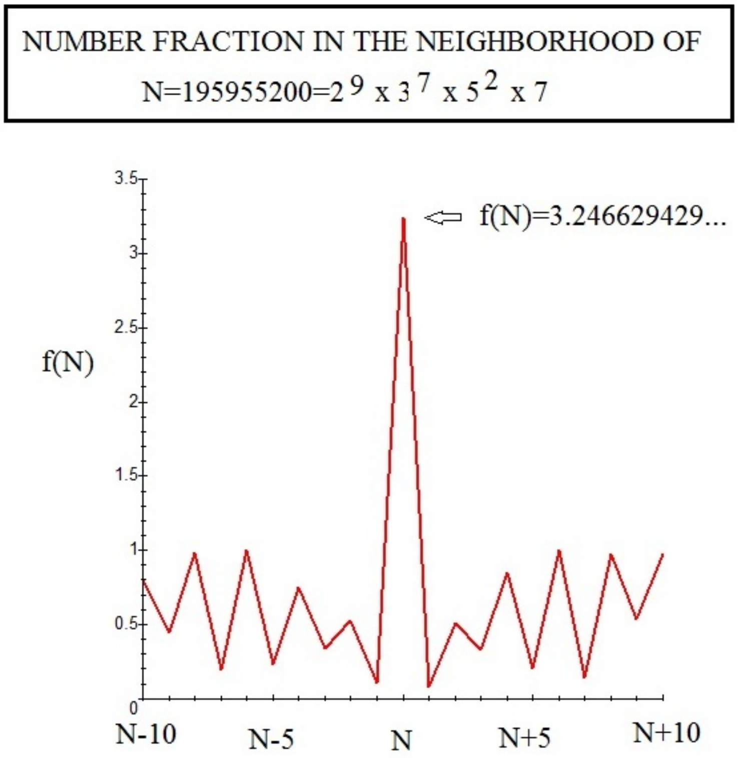
680	2^4x3x5x7	2.542261905…
1260	2^2x3^2x5x7	2.465873016…
1680	2^4x3x5x7	2.542261905…
2520	2^3x3^2x5x7	2.713888888…
5040	2^4x3^2x5x7	2.837896825…
90720	2^5x3^4x5x7	3.033322310…

From the table we see that local maxima in f(N) occur when the number N is expressible as the product of the lowest primes taken to descending powers. So the multiple of 12 observed between local maxima of f(N) in the first figure above stems from the fact that all those numbers contain 2^2x3. A slow increase in the local maxima in f(N) with increasing N is noted.

With this information let us look at the number-

$$N=2^9x3^7x5^2x7= 195955200$$

One suspects this represents a super-composite as the following graph confirms--



Note that for this nine digit long number the value of f(N) has only risen to a value of 3.25. Also here there are no primes found within ten units on either side of N. However six primes are found when the range extends 20 units on either side of N. The average value of the f(N)s remains low at about 0.5.

In searching for ever larger values for local maximums of f(N), we have found in view of earlier data that a convenient number generator for super-composites is the number defined by the products of ascending primes taken to the descending powers of primes. This new number can be defined as-

$$N[p_m]=\prod_{k=1}^{m}(p_k)^{m+1-k}\ =\ p_1{}^{m}+p_2{}^{m-1}+.....+p_m{}^{2}\quad where$$
$$p_1=2,\ p_2=3,\ p_3=5,\ p_4=7,\ p_5=11\quad etc$$

Its size increases very rapidly with increasing m and always will end in zeros when m is 3 or greater. Its first ten values are given by the following jpg-

FIRST TEN VALUES OF THE EVEN NUMBER $N[p_m]$

{1, 4}
{2, 72}
{3, 21600}
{4, 190512000}
{5, 580909190400000}
{6, 428616352408083840000000}
{7, 859278392084450410309036800000000000}
{8, 2097197194438629126172451944256706311040000000000000}
{9, 6647256211429027050577887108708833812953545505219891840000000000000000}
{10, 14425379128610056061564609695053863929767057289202007928598639762339728309248000000000000000000000}

p_1=2 , p_2=3, p_3=5, p_4=7, p_5=11, p_6=13, p_7=17, p_8=19, p_9=23, p_{10}=29

Note that the square bracket N[ ] notation we are using to describe this number differs from the standard functional form N( ). The number N[ ] which defines a product function is generally much larger than is a number N( ).

At m=4 corresponding to p_4=7 we have-

$$N[p_4]=2^7\cdot 3^5\cdot 5^3\cdot 7^2=190512000$$

It has f(N)= 3.332301582. The nearest primes occur for N-11 and N+17. That is, 190511989 and 1905112017 are primes. Consider next the larger product number-

$N[p_{12}]=\ 2^{37}\cdot 3^{31}\cdot 5^{29}\cdot 7^{23}\cdot 11^{19}\cdot 13^{17}\cdot 17^{13}\cdot 19^{11}\cdot 23^{7}\cdot 29^{5}\cdot 31^{3}\cdot 37^{2}$

=7524033990120088901785873974521982669687764716167188342113040120
5241979583045287483491616420091948692672897265719856123802549998080000000000000000000000000000 .

Its number fraction reads f(N)= 5.723859385… and so $N[p_{12}]$ is a super-composite. The closest prime found for this 160 digit long number is at N-41. Taking our home PC to its calculation limit (corresponding to m=100), we find f=10.26762032.. The value of N corresponding to this m is several pages long and will not be printed out here.

We are not sure yet if $N[p_m]$ has a finite or infinite value at a maximum for the corresponding number fraction f(N) as N goes to infinity. However the last two results would seem to support the conjecture that a maximum in f(∞) should become infinite. Most other numbers, not at a local maximum, will have f(N)s lying below unity. We will demonstrate this below.

Let us next look at some of these f(N) values away from local maxima. One such set of numbers is $N=2^n$. Here we have-

$N=2^n$	2	4	8	16	32
f(N)	0	1/2	3/4	7/8	15/16

From this, one sees at once that-

$$f(2^n)=\frac{1}{1}\{1-\frac{1}{2^{n-1}}\}$$

This means that f(64)=31/32 and also that $f(2^n)$ goes toward one as n becomes infinite. Trying the same generalization for powers of 3 we find-

$$f(3^n)=\frac{1}{2}\{1-\frac{1}{3^{n-1}}\}$$

and for powers of the next prime we have-

$$f(5^n)=\frac{1}{4}\{1-\frac{1}{5^{n-1}}\}$$

An overall generalization for all primes then yields-

$$f(p^n)=\frac{1}{(p-1)}\{1-\frac{1}{p^{n-1}}\}$$

This last result shows that the number fraction for p^n approaches the very small value of 1/(p-1) as n gets large.

The above relation for p^n does not hold when N is a composite. However it is still easy to find f(N) in cases such as where N is a semi-prime N=pq . One has-

$$f(pq) = \frac{p+q}{pq}$$

This means, for example, that-

$$f(77) = f(7x11) = \frac{18}{77} = 0.233766...$$

and-

$$f(77851) = f(127x613) = \frac{740}{77851} = 0.009505337...$$

Note that f(N) for semi-primes will remain just slightly above zero as N gets large. Only pure primes will have f(p)=0.

Continuing on, we find that the triple prime N=pqr has its number fraction given by-

$$f(pqr) = \frac{(p+q+r)+(pq+pr+qr)}{pqr}$$

Thus f(1771)=f(7x11x23)=[41+(77+161+253)]/1771=76/253=0.300395…

Again the typical value of f(N) for a triple prime is less than one.

One can construct formulas for numbers such as $N=p^n x q^m x r^k$, but these become rather cumbersome. For numbers of this type it is much simpler to find f(N) using the definition formula given at the beginning of this article directly. A number such as-

N= 1234681423= (17) (23) (31) (101863)

is expected to not be a super-composite. Carrying out a calculation for its number fraction, we find f(N)=0.1405111543… .

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