PROBABILITIES ASSOCIATED WITH COIN, MARBLE, AND DICE GAMES

It is well known that the simplest game of chance involves the flipping of a single coin. There are just two-possible outcomes, namely, heads(H) or tails(T). For a properly balanced and flipped coin the probability for each of the two outcomes is p= 50%. These numbers will change if more than one coin is involved. For example flipping two identical coins simultaneously can produce any-one of the following four possible outputs (H+H), (H+T), (T+H), and (T+T). at a probability of p=25% each. We wish here to examine the outcome probabilities of flipping multiple coins , withdrawing marbles from an opaque jar, and rolling multiple die.

Let us begin with coin tossing with n coins at a time. The outcome for the first three cases can be summarized as follows-

Coins Involved	Possible Outcomes
1	[H,T]
2	[H,H],[H,T],[T,H],[T,T]
3	[H,H,H],[H,H,T],[H,T,H],[T,H,H],[H,T,T],[T,H,T],[T,T,H],[T,T,T]

.
If one looks at the progression of outcomes it is clear that the number of distinct outcomes equals 2^n with n being the number of coins involved. Sine the probability of each outcome is equally likely, the chance of getting all heads in a three coin flip is just p=1/8=12.5%.The chance of getting two heads and one tail or two heads and one tail is three times larger and equal to p=3/8=37.5%. To verify the 2^n rule consider next the case of flipping 4 coins simultaneously. Here we find the 16 combinations-

[T,T,T,T],
[T,T,T,H],[T,T,H,T],[T,H,T,T],[H,T,T,T],
[T,T,H,H],[T,H,H,T],[T,H,T,H],[H,T,H,T],[H,T,T,H],[H,H,T,T],
[T,H,H,H],[H,T,H,H],[H,H,T,H],[H,H,H,T],
[H,H,H,H]

Placing their numbers in a row we have $1+4+6+4+1=2^4$. Aligning a ll of the above coin toss possibilities in an array , we find-

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            1
        1   2   1
     1    3    3    1
  1    4    6    4    1
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This represents the famous Pascal Triangle. The next row, corresponding to flipping five coins simultaneously, predicts that the probability of five tails or five heads coming up has the very low probability of p=1/32=3.125%. The coefficient in the mth row and nth column is given by the binomial coefficient C[m,n]=(m+1)!/(n+1)!(m-n)!). Here we count the first row as row 1 and the first column as column 1. So the number of four component groups with exactly two heads and two tails coming up in a four coin flip will be exactly C[4,3]=5!/(4!·1!)=6.The probability of this combination coming up will be p=6/32=18.75%.

Probabilities associated with coin flipping are identical with picking marbles out of an opaque jar containing well mixed half white and half black marbles. The probability of picking either a white or black marble on the first try is 50-50. To pick out 4 black marbles in a row is slightly less than $(1/2)^4$=1/16. The slightly smaller value stems from the fact that the black number in the jar will become slowly depleted as more black marbles are picked than predicted. Playing Roulette using the black-white or even-odd option can also be related to coin flipping. In standard American roulette there are 16 red and 16 black pockets together with two green pockets. Thus during one rotation of the roulette wheel the chance of hitting red or black is p=16/38=42.1% each, while the chance of hitting green is 2/38=5.25%. Since hitting a green pocket means a house win, one can say that the house advantage is 5.25% guaranteed in the long run. It is of advantage for a gambler to play their roulette in Europe where the house advantage comes from just one green pocket meaning that the house gain is just 1/37=2.70% per wager.

A two marble game of chance can be constructed by placing an equal number of black(1), white(2) marbles into an opaque jar and mixing them well. If one next withdraws just a single marble the probability of getting either one or the other of the two colors is p=1/2=50%. If we change things to withdrawing two marbles at a time there will be the possibility of four combinations which are [1,1], 1,2], [2,1], and [2,2]. At first glance the numbers [1,2] and [2,1] would appear to be identical until one realizes that pulling two marbles at a time is equivalent to pulling one marble at a time in succession. Thus the result [1,2] is not the same as [2,1] although their face value sum are the same. The probability of getting two black marbles on the first trial is 1 in 4 and not getting the two black marbles is 3 to 4. The chance of success is thus 1/4=25% and that of failure is p=75%. It will often be convenient when dealing with such multiple withdrawal games to put things in tabular form as follows-

Face Value , S	2	3	4
Possible Combimations , T	[1,1]	[1,2] [2,1]	[2,2]
Meaning:	two black	one black one white	two white

With such a numerical identification of marble colors it is now a simple matter to see what happens with n different color marbles. For a three color problem (black=1, white=2, and red=3) we get the table-

S=	3	4	5	6	7	8	9
T=	[1,1,1]	[1,2,1]	[1,3,1]	[1,3,2]	[1,3,3]	[2,3,3]	[3,3,3]
		[1,1,2]	[1,1,3]	[1,2,3]	[2,2,3]	[3,2,3]	
		[2,1,1]	[1,2,2]	[2,1,3]	[2,3,2]	[3,3,2]	
			[2,1,2]	[2,3,1]	[3,1,3]		
			[2,2,1]	[2,2,2]	[3,3,1]		
			[3,1,1]	[3,1,2]	[3,2,2]		
				[3,2,1]			

Here [1,1,1], [2,2,2], and [3,3,3] represent three black, three white, and three red marbles. There are a total of 27 combinations shown in the table indicating that to get three white marbles on the first three marble withdrawal, the probability is just p=1/27=3.70%.

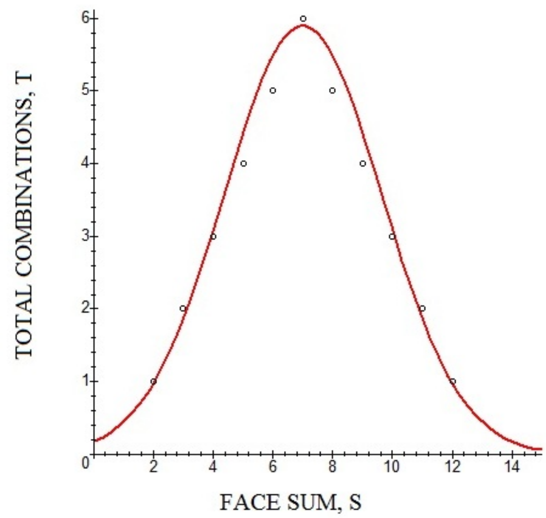
In addition to games involving just two outcomes and /or extensions thereof, one has games involving dice where each dice offers six possible outcomes. That is, if one rolls a single dice, the outcome is 1/6 for each of the six numbers 1,2,3,4,5,or 6. So the probability of hitting any of these six numbers will be the same at p=16.667%. Playing Russian Roulette represents such a six-number game since one places one live round into a revolver , spins the cylinder , and then points it toward one’s head and pulls the trigger. The chance of being killed will be 16.67% and not being killed will be 83.33%. Repeated playing of the game does not change the probability until a point is reached where ones number comes up. Perhaps the best known two dice game is craps where the probability of getting snake eyes(S=2) or boxcars(S=12) is 1 in 36 . An S-T table for craps follows-

Sum,S:	2	3	4	5	6	7	8	9	10	11	12
Possible	[1,1]	[1,2]	[1,3]	[1,4]	[1,5]	[1,6]	[2,6]	[3,6]	[5,5]	[6,5]	[6,6]
Combos		[2,1]	[3,1]	[4,1]	[5,1]	[6,1]	[6,2]	[6,3]	[6,4]	[5,6]	
,T			[2,2]	[2,3]	[2,4]	[2,5]	[3,5]	[4,5]	[4,6]		
				[3,2]	[4,2]	[5,2]	[5,3]	[5,4]			
					[3,3]	[3,4]	[4,4]				
						[4,3]					

There are now a total of N=36 combinations with the probability 1/36 =2.778% of any one of these occurring. There are six combinations which return a sum of seven and only one possibility for S=2. Thus one can say that the ratio of snake eyes compared to a sum of S=7 coming up is one in seven. The true probability for the [1,1] combo is just 1/36=2.778%. Note the symmetry in the number of combinations T about the mean sum of $\bar{S}$ = (2+12)/2=7.That is, the three combos with the sum of S=4. has its counterpart as three combos with a sum of S=10.

Note also the near Gaussian shape for the values of T in the last table. This is not a coincidence but represents a limit which will be reached if one increases the number of die used at any one time. A point graph of the number of combinations T versus the face values S for the two die case (such as in craps) follows-

COMBINATIONS (T) VERSUS FACE SUM (S) FOR A TWO DIE ROLL

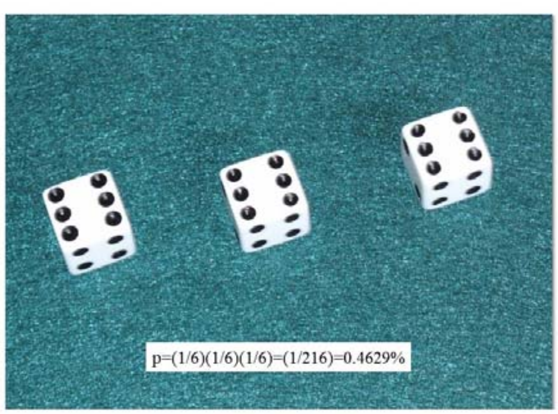


Gaussian Estimate $T=5.9\exp(-0.072(S-7)^2$

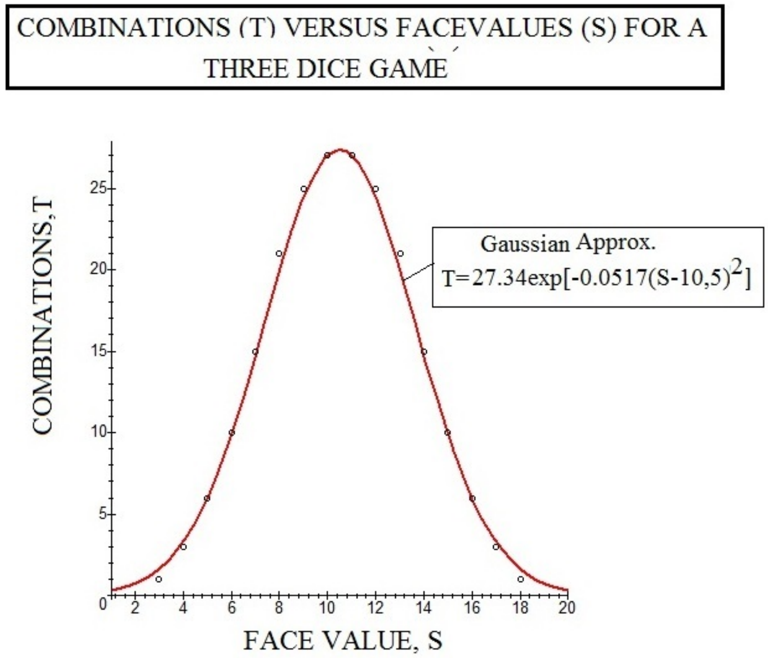
There we have sketched in a Gaussian curve in red as opposed to the exact integer T versus S values represented by small circles.

Finally we could consider a three dice game where the chance of three sixes coming up simultaneously should be $(1/6)^3=1/216=0.4629\%$. A picture of such an outcome follows-

STAGED TRIPLE BOXCARS



Also we show in the following graph the relation of T versus S for such a three dice game-



There are a total of 216 combinations found of which only six are triplets. The probability of their appearance should be 6/216 per throw. I have carried out a three dice experiment tossing three die simultaneously for a total of 216 times. The results were 8 triplets (3 all ones, 1 all twos, 2 all fours,1 all fives,1 all sixes). This experimental result lies close to the expected value of 216/36=6 triplets. Note the excellent fit of a Gaussian to the T versus S results.

We also earlier performed a two penny flip test. After 44 simultaneous flips with two pennies the results yielded 10 double Ts, 22 (H+T)s, plus 12 double Hs. This result is right on the prediction that the number of (T+T)s and (H+H)s represents 50% of the results. People playing the lottery are aware of this fact.They know that if a single ticket has a one in a one-hundred million chance of winning, buying a hundred tickets should increase the odds to one win in one million. Still not very good odds, but then 100 times better than just buying one ticket. Those individuals not participating obviously have no chance of winning although they clearly save the price needed for tickets.

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