PROPERTIES OF THE GAUSSIAN FUNCTION

The Gaussian in an important 2D function defined as-

$$y(x) = a\exp - \{\frac{(x-b)^2}{c}\}$$

, where a, b, and c are adjustable constants. It has a bell shape with a maximum of y=a occurring at x=b. The first two derivatives of y(x) are-

$$y'(x) = -\frac{2a(x-b)}{c}\exp - \{\frac{(x-b)^2}{c}\}$$

$$and$$

$$y''(x) = \frac{2a}{c^2}\left\{-c + 2(x-b)^2\right\}\exp - \{\frac{(x-b)^2}{c})$$

Thus the function has zero slope at x=b and an inflection point at x=b±sqrt(c/2). Also y(x) is symmetric about x=b. It is our purpose here to look at some of the properties of y(x) and in particular examine the special case known as the probability density function.

Karl Gauss first came up with the Gaussian in the early 18 hundreds while studying the binomial coefficient C[n,m]. This coefficient is defined as-

$$\text{C[n,m]=} = \frac{n!}{m!(n-m)!}$$

Expanding this definition for constant n yields-

$$C[n,m] = n!\left\{\frac{1}{0!(n-0)!} + \frac{1}{1!(n-1)!} + \frac{1}{2!(n-2)!+} + \ldots \frac{1}{n!(0)!}\right\}$$

As n gets large the magnitude of the individual terms within the curly bracket take on the value of a Gaussian. Let us demonstrate things for n=10. Here we have-

C[10,m]= 1+10+45+120+210+252+210+120+45+10+1=1024=2^{10}

These coefficients already lie very close to a Gaussian with a maximum of 252 at m=5. For this discovery, and his numerous other mathematical contributions, Gauss has been honored on the German ten mark note as shown-

If you look closely, it shows his curve.

The Gaussian contains certain built-in length dimensions. These include its height at x=b, the distance from x=b to the two symmetrically located inflection points and the area under the curve. Only two of these dimensions are needed to specify the values of a and c. The b can be chosen to begin with. It centers the curve at x=b. When b=0 the Gaussian is symmetric about x=0. the area under the Gaussian can readily be calculated . It equals-

$$Area = a\int_{x=-\infty}^{\infty} \exp-\{\frac{(x-b)^2}{c}\}dx = a\sqrt{\pi c}$$

If we now let the area be unity and let the distance from x=b to the inflection points be defined as the standard deviation σ, we have –

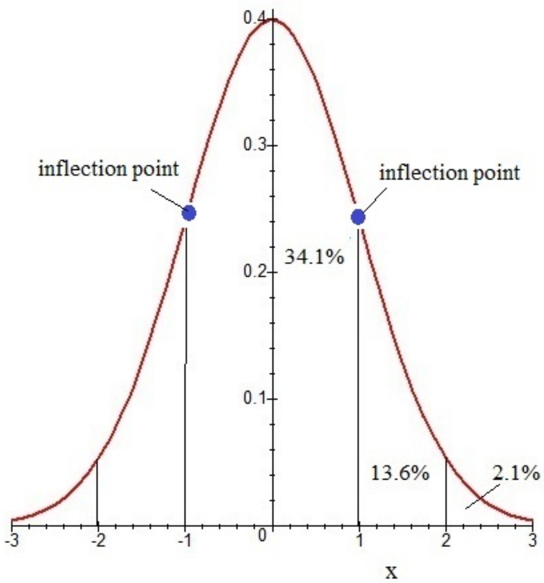
$$a = \frac{1}{\sigma\sqrt{2\pi}} \quad and \quad c = 2o^2$$

Also we change b to μ and call this distance from the origin the Gaussian mean. Substituting these constants into the original Gaussian we arrive at the Probability Density Function-

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp-\{\frac{(x-\mu)^2}{2\sigma^2}\}$$

It plays a major role in statistics and also finds applications in areas such as the temperature along a long bar away from a local hot spot. In addition it enters the discussion on the IQ distribution of a group of individuals. Setting the standard deviation to σ=1 and letting μ=0, produces the famous Bell curve Z(x) -

THE BELL CURVE Z(x)={1/sqrt(2 π) }exp-(x ² /2)



The total area under this curve remains at one and each digit increase in x represents an extra standard deviation. The inflection points on the graph lie at x=±1. The number of individuals with an IQ lying between x=0 and x=1 is given by the fraction-

$$\frac{1}{\sqrt{2\pi}}\int_0^1 \exp-(\frac{x^2}{2})=\frac{1}{2}erf(\frac{1}{\sqrt{2}})=0.3413447..$$

of the population. The fraction lying between the (n-1) and (n)th standard deviation will be-

$$\frac{1}{2}\left\{erf(\frac{n}{\sqrt{2}})-erf(\frac{n-1}{\sqrt{2}})\right\}$$

So the first few values , given in table form, become-

N	1	2	3	4
Fraction	0.341344	0.135905	0.0214002	0.00131822

These values add up to 0.499967 meaning that the total area under the remaining values for positive x add up to a miniscule amount . The error function appearing in the solution is defined as-

$$erf(x)=\frac{2}{\sqrt{\pi}}\int_{t=0}^{x}\exp(-t^2)dt$$

Note that an individuals IQ score is set at 15 points above or below 100 for each standard deviation. Thus an IQ of 145 means three standard deviations to the right of the bell curve maximum. To become a member of MENSA requires an IQ score of at least 132.That is, only about-

$$1-\frac{1}{\sqrt{2\pi x}}\int_{x=-\infty}^{32/15}\exp-\{\frac{x^2}{2}\}dx=1.64\%$$

of the population is eligible for membership.

There are numerous relationships involving the Bell curve Z(x) and the related Probability Density Function P(x). Many of these are found in the Abramowitz and Stegun, "Handbook of Mathematical Functions" available free online. Let us look at a few of these here.

Start with the infinite series expansion of Z(x) about x=0. We have-

$$Z(x)=\frac{1}{\sqrt{2\pi}}\int_{t=0}^{x}\exp-(\frac{t^2}{2})dt=\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!2^n(2n+1)}$$

This result is obtained by expanding $\exp(-t^2/2)$ in a taylor series about t=0 and then integrating. If we now write out Z{x) and its first two derivatives in series form, we get-

$$Z(x)=\frac{1}{\sqrt{2\pi}}\left\{1-\frac{1}{2}x^2+\frac{1}{8}x^4-\frac{1}{48}x^6+...\right\}$$

$$xZ'(x)=\frac{1}{\sqrt{2\pi}}\left\{-x^2+\frac{1}{2}x^4-\frac{1}{8}x^6-...\right\}$$

$$Z''(x)=\frac{1}{\sqrt{2\pi}}\left\{-1+\frac{3}{2}x^2-\frac{5}{8}x^4+\frac{7}{48}x^6-...\right\}$$

Next adding up the three series terms, yields the seconds order differential equation-

$$\frac{d^2Z}{dx^2}+x\frac{dZ}{dx}+Z=0$$

If we integrate P(x) over the range 0<x<∞, one finds-

$$\int_{x=0}^{\infty} P(x)dx = \frac{1}{2} = \frac{1}{\sigma\sqrt{2\pi}}\int_{x=0}^{\infty} \exp(-\{\frac{(x-\mu)^2}{2\sigma^2}\} = \frac{1}{2\sqrt{\pi}}\int_{x=0}^{\infty} \frac{\exp(-u)}{\sqrt{u}}du$$

Recognizing that the last integral is just the Laplace transform of $t^{(-1/2)}$ with s=1, we obtain the identity-

$$\sqrt{\pi} = \Gamma(\frac{1}{2})$$

, with Γ(1/2) being the gamma function of ½.

Finally, there are numerous integrals involving the Gaussian. Among these we have-

$$\int_{t=\mu}^{x} P(t)dt = \frac{1}{2}erf\left\{\frac{(x-\mu)}{\sigma\sqrt{2}}\right\}$$

$$\int_{x=0}^{\infty} \cos(x)Z(x)dx = \frac{1}{2\sqrt{\exp(1)}}$$

and

$$\int_{x=0}^{\infty} x^n Z(x)dx = \frac{\Gamma(\frac{n+1}{2})}{2\sqrt{\pi}(\frac{1}{2})^{1/2}}$$