DISCOVERING MATHEMATICAL PRINCIPLES BY GENERALIZATION

In mathematics the sub-field of Number Theory lends itself to more than almost any other discipline to generalizations following from specific examples. We want here to demonstrate this fact by starting with the set of all positive integers and then see what type of relations of a general nature one can be deduce from a limited number of specific examples.

Our starting point is the set of integers-

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 …n...

The even integers are all those numbers in this group which are divisible by 2. Thus all even integers can be written as-

2 4 6 8 10 12 14 16 18 20 …2n…

The remaining integers are the odd numbers-

1 3 5 7 9 11 13 15 17 19…2n-1…

Some of these odd integers are divisible only by 1 and themselves. These are the prime numbers-

2 3 5 7 11 13 17 19…p…

With the exception of 2, these prime are all odd numbers.

Lets first consider the sum of all even integers up to 2n. We have the specific cases-

$2+4=6=2^2+2$

$2+4+6=3^2+3$

$2+4+6+8=4^2+4$

$2+4+6+8+10=5^2+5$

From these we can generalize things to-

$$\sum_{k=0}^{n} 2k = 2+4+6+...+2n = n^2+n = n(n+1)$$

Next dividing both sides of this equality by 2 produces the well-known result-

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

If we want to add up only the odd numbers, we get-

$$\sum_{k=1}^{n} (2k-1) = 1 + 3 + 5 + ... + (2n-1) = n(n+1) - n = n^2$$

So adding up the first 100 odd numbers we have 10,000.

Next let us see what can be done with the sum of the squares up to N^2. We have the special cases-

$1^2+2^2=5$

$1^2+2^2+3^2=14$

$1^2+2^2+3^2+4^2=30$

$1^2+2^2+3^2+4^2+5^2=55$

In view of the form of the sum of the first N integers, one suspects that the sum of the squares will have the form of a cubic-

$$\sum_{k=1}^{N} k^2 = aN^3 + bN^2 + cN + d$$

,where the numbers a, b, c, and d are to be determined using the above specific results. Using these sums one obtains the matrix equation-

$$\begin{bmatrix} 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \\ 125 & 25 & 5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 5 \\ 14 \\ 20 \\ 55 \end{bmatrix}$$

On solving this equation, one finds a=1/3, b=1/2, c=1/6 and d=0. Thus we have-

$$\sum_{k=1}^{n} k^2 = \frac{n(2n+1)(n+1)}{6}$$

This means that the sum of the squares of the first ten integers equals 385. There is no major difficulty in extending the discussion to the mth power of the first N integers using

the same argument as above but based upon the specific cases of the first m algebraic equations. The sum of the first n third powers behave as follows-

$1^3 + 2^3 = 3^2$

$1^3 + 2^3 + 3^3 = 6^2$

$1^3 + 2^3 + 3^3 + 4^3 = 10^2$

$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 15^2$

We can generalize this as- $$\sum_{n=1}^{N} n^3 = \left(\sum_{n=1}^{N} n \right)^2 = \left[\frac{n(n+1)}{2} \right]^2$$

Thus the sum of the cubes of the first 100 integers equals 25502500.

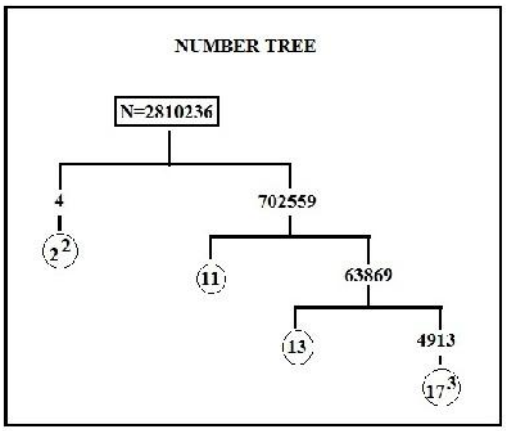
One notices that all those integers which are not primes can always be expressed as the products of primes taken to specified powers. Thus we have-

$4=2^2$, $6=2^1x3^1$, $8=2^3$, $9=3^1x3^1$,$10=2^1x5^1$, $12=2^2x3^1$, and $14=2^1x7^1$

That is , any number N can be written as-

$$N = \sum_{k=1}^{m} {p_k}^{a_k}$$

, where p_k is the kth prime , a_k its exponent, and m the maximum prime required. The first seven primes are $p_1=2$, $p_2=3$, $p_3=5$, $p_4=7$, $p_5=11$, $p_6=13$, and $p_7=17$. A larger number such as 2810236 can be expressed as $2^2x11^1x13^1x17^3$. A simple way to find the prime number product of such numbers is to use a number tree as follows-



One notices that the exponents shown in a prime number expansion of a number is unique to that number. Hence we can introduce an exponent vector –

$$V=[a_1,a_2,a_3,..]$$

to describe N. The vector for N=2810236 becomes V=[2 0 0 0 1 1 3]. One of the properties that an exponent description of a number makes possible is to quickly find what number N_2 is required to make the product of N_1 x N_2 a perfect square. Take the case of the number $N_1=572=2^2$ x 11^1 x 13^1. What number N_2 will make their product a perfect square? The answer is $N_2=11^1$ x$13^1=143$. So we have that 81796=572 x 143 has the sqrt(81796)=286. There are of course an infinite number of other N_2s which will make the product a perfect square. All that is required that the exponent vector of the product N_1xN_2 have only even elements. Another advantage of the exponential expression for numbers is that it quickly allows one to find the greatest common denominator gcd of two numbers. Take $N_1=312=2^3\cdot3^1\cdot13^1$ and $N_2=234=2^1\cdot3^2\cdot13^1$. Here the common factor is $2\cdot3\cdot13=78$. Hence-

$$\gcd(312, 234)=78$$

We next ask what is the product of the first N integers . It is simply N factorial denoted by N! . Thus-

$$5!=\sum_{k=1}^{5}k = 1x2x3x4x5=2^3\cdot3\cdot5=120 \quad \text{and} \quad 12!=\prod_{k=1}^{12}k =2^{10}\cdot3^5\cdot5^2\cdot7^1\cdot11^1=479001600$$

Note that N!(N+1)=(N+1)! and N!(N+1)(N+2)=(N+2)!. Generalizing we have-

$$\prod_{k=1}^{m}(N+k)=\frac{(N+m)!}{N!}$$

On setting m=1, we have the generating formula-

$$(N+1)!=N!(N+1)$$

and on setting m=N we have-

$$(2N)!=N!\prod_{k=1}^{N}(N+k)=N!\{(N+1)(N+2)(N+3)...(N+N-1)(2N)\}$$
$$=N![2N^2+2N][2N^2+3N-2][2N^2+4N-6][2N^2+5N-12][...$$
$$=N!\prod_{k=1}^{N/2}[2N^2+(k+1)N-k(k-1)]$$

This last result works for even N but must be modified if N is odd. As an example, we have –

$$8!=4!\{(32+8-0)(32+12-2)\}=24(1680)=40320$$

If we have two numbers a and b it is possible to take their sum to the nth power. For the first few ns this produces-

$(a+b)^2=a^2+2ab+b^2$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$

$(a+b)^4=a^4+4a^3b+6a^2b^2+3ab^3+b^4$

We can generalize these results to obtain the well-known Binomial Formula-

$$(a+b)^n=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}a^{n-k}b^k$$

The factorial term in this expression is the Binomial Coefficient often denoted by C[n,k] . The coefficient also defines the elements of a Pascal Triangle.

If we look at the sum of all divisors of a number N we obtain what is known as the sigma or divisor function σ(N). For N=24 this yields σ(24)=1+2+3+4+6+8+12+24=60. A better summation introduced by us several years ago is the number fraction function f(N) defined as –

$$f(N)=\frac{\pm(N)-N-1}{N}$$

We designed this function so that f(N)≡0 for all primes N=p. That is, for any prime the only divisors are 1 and p so that the numerator of f(p) will always vanish. Furthermore the function has the important property that-

$$f(p^n)=\frac{1-p^{1-n}}{p-1}$$

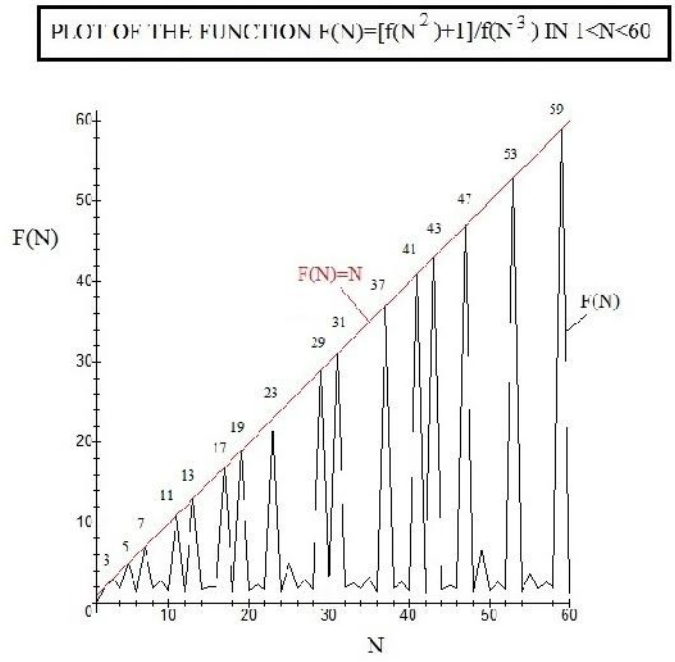
This means that $f(p^2)=1/p$ and $f(p^3)=(1+p)/p^2$. Solving for p we get-

$$p=\frac{[f(p^2)+1]}{f(p^3)}$$

This can be thought of as a criterion for N being a prime p. If we take the number N=7 we get $f(7^2)=1/7$ and $f(7^3)=8/49$ and the quotient reads (1/7+1)/(8/49)=7. So 7 is a prime. On the other hand if we look at N=9 we get $f(9^2)=13/27$ and $f(9^3)=121/243$. This means the quotient equals 360/121 which differs from 9. Hence N=9 is a composite number. Of course the same conclusion can be reached by noting f(9)=1/3 and so does not vanish. An interesting function follows from the above. We term it the **Prime Number Function** and define it as-

$$F(N) = \frac{f(N^2)+1}{f(N^3)}$$

Whenever N is a prime, F(N) =N while for all other integers the quotient will be less than N. This fact is nicely shown in the following graph-



Note that the function has local maxima only when N is a prime number.

That a generalization in number theory is valid for all N may not always be so. For example, the French cleric Mersenne back in the 16 hundreds observed that-

2^2-1=3 , 2^3-1=7 , 2^5-1=31, 2^7-1=127

So he concluded that the function 2^p-1 always produces primes. Unfortunately this conjecture was proven to be wrong as already shown by the next number 2^{11}-1=2047 which factors as 23 x 89. Indeed only some 48 cases of primes of the Mersenne type have been found to date. Likewise , Fermat observed that-

$2^{2^n}+1$ yields primes for n=1, 2, 3, and 4

He thought this observation should hold for 5 and higher. It was Euler who first showed that 2^{32}+1 is not prime but rather a composite 641 x 6700417. To date no one has been able to find a Fermat prime for n=5 or higher yet no proof exists that this is true for all n. Such negative results of generalization in number theory might call into question the utility of generalization. This however is not the case provided one confirms the results

for a large number of cases. Under those conditions the probability that the generalization is valid produces a very high level of confidence.

Let us complete the present discussion by seeing if we can come up with some original observations in number theory based on generalization. On writing down the primes in ascending order starting with p=5 we note the following-

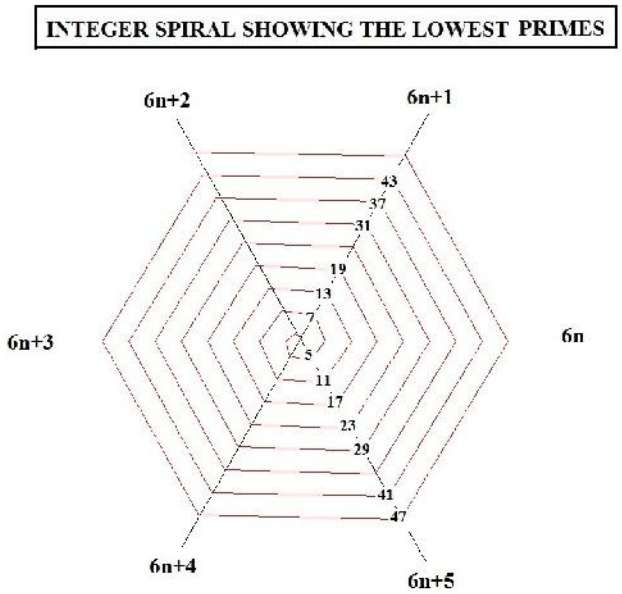
5=6(1)-1 , 7=6(1)+1 , 11=6(2)-1 , 13=6(2)+1 , 17=6(3)-1 , 19=6(3)+1 , 23=6(4)-1
29=6(5)-1 , 31=6(5)+1 , 37=6(6)+1 , 41=6(7)-1 , 43=6(7)+1, and 47=6(8)-1

From these results we can generalize and state that-

All primes above N=3 have the form 6n±1

We find no exceptions to this conjecture although there are also composites which have the form 6n±1. To check whether we are missing any primes not of the form 6n±1, we have gone to our computer and observe that the p_3=5 and p_{15} =47. So that the number of primes in the range 5 to 47 equals 15-3+1=13. This checks exactly with the number of primes in the above list, meaning there are no primes other than of the type 6n±1. Thus an odd number such as N=6(33156)+3=198939 can never be a prime but N=6(64333)-1=385997 might be.

Furthermore the fact that if one thinks of the numbers 6n +k , for k=0, 1, 2, 3, 4, and 5, as lying along radial lines at π/3 radian intervals from each other when measured in polar coordinates, one gets the following pattern-



A hexagonal spiral with corners at all positive integer values has been superimposed. Note that all the primes fall along the straight lines 6n+1 or 6n-1 and thus don't show the randomness for primes found with a Ulam spiral (see our 2008 article at–

http://www2.mae.ufl.edu/~uhk/MORPHING-ULAM.pdf.). Note that 6n+5 is equivalent to 6n-1 and that 6n+1 numbers always have N mod(6)=1 and 6n-1 numbers have N mod(6)=5. Double primes are here characterized by lying along the same turn of the spiral and symmetric about the 6n axis. Thus 5-7, 11-13, 17-19, 29-31, and 41-43 are all examples of double primes.

Since all primes above p=3 have been shown to be of the form 6n±1, it stands to reason that one should be able to quickly find very large primes by simply defining a random large number x=6n±1 and finding those values for which N=x+6m fields f(N)=0.This search can be done by computer using the one line command-

N:=x+6*m; for m from 1 to 50 do {m, evalf(f(N)}od;

Let us demonstrate by looking at a random number which may or not be a prime-

x:=6*(678430121731213908546324572 8)-1= 407058073038728345127794743 67

Here one finds f(N)=0 for m=11, 34,… . Thus one has the primes-

407058073038728345127794744 33

and

407058073038728345127794745 71

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