GENERATING A PRIME NUMBER FUNCTION

We have shown in several earlier notes that any positive integer has associated with it a number fraction:

\[ f(N) = \frac{\{\text{sum of the divisors of } N - (N + 1)\}}{N} = \frac{(\sigma(N) - (N + 1))}{N} \]

, where \( \sigma(N) \) is the sigma function of number theory. The function \( f(N) \), which is a rational fraction, has the important property that it vanishes if and only if \( N \) is a prime number.

Proceeding to higher powers of the primes \( p \), we have:

\[ f(p^2) = \frac{1}{p} \]
\[ f(p^3) = \frac{1 + p}{p^2} \]
\[ \text{and} \]
\[ f(p^4) = \frac{1 + p + p^3}{p^3} \]

From these results one can conclude that:

\[ f(p^n) = \sum_{k=1}^{n-1} \frac{p^k}{p^{n-k-1}} = \frac{1 + p + p^2 + \ldots + p^{n-2}}{p^{n-1}} \]

Thus \( f(121)=1/11, f(625)=f(5^4)=31/125, \) and \( f(343)=f(7^3)=8/49 \) etc. It is also noted that one can manipulate the above results to obtain:

\[ 1 = \frac{1}{p} + \frac{1}{p^2} \]
\[ \frac{f(p^2) + 1}{pf(p^3)} \]

, provided that \( p \) remains a prime number. So for \( p=7 \), we find \( \{7(8/49)-(1/7)\}=1. \) If we now introduce a new function:

\[ F(N) = \frac{f(N^2) + 1}{Nf(N^3)} = \frac{[\sigma(N^2) - 1]}{[\sigma(N^3) - 1] - N^3} = \frac{g(N^2)}{g(N^3) - N^3} \]

, it is clear that \( F(N) \) will always be equal to one if \( N=p \). However, when \( N \) is not a prime then \( F(N) \) will typically be less than one such as \( F[6]=(f(36)+1)/(6f(216))=90/283 \) and \( F(8)=21/85. \) The prime \( p=7 \) yields \( F(7)=1. \) The function \( g(x) \) is defined as \( g(x)=\sigma(x)-1 = s+x-1, \) where \( s \) is the aliquot sum which sums all divisors of \( N \) except the last at \( x=N. \) The
function $F(N)$ has meaning in the strict sense only if $N$ is an integer for only then will the number fraction $f(N)$ yield a unique rational value. However we can approximate $F(N)$ as a continuous function $F(x)$ (but jumps in its derivatives remain) by introducing a linear interpolation of $F(x)$ between $N$ and $N+1$ as shown-

Such an interpolation allows one to plot $F(x)$ as a continuous function over any desired range. In computer language, using our MAPLE program, we can write-

```
with(plots):
with(numtheory):
F:=\sigma(x^2)-1)/((\sigma(x^3)-x^3-1):
listplot([seq([x,F],x=3..50)])
```

It produces the following graph-
over the range 3<x<50. The peaks at F(x)=1 represent the primes which show up very clearly. What is nice about this function is that it will work over any desired range within the limits of ones PC’s ability to find f(x^2) and f(x^3). One can easily print out the numbers x for which f(x)=0 and F(x)=1. These will be the primes. For example the one line program–

for x from 980 to 1000 do {x, f, F} od;

produced in a split second the three primes-

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)={σ(x)-x-1}/x</th>
<th>F(x)={σ(x^2)-1}/(σ(x^3)-x^3-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>983</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>991</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>997</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

in the range 980<x<1000. As this table shows, one can either have f(x)-0 or F(x)=1 for x to be a prime. The f(x)=0 criterion is somewhat faster to evaluate, but for graphic representation the F(x)=1 condition is preferred.

We can use the above results to discuss the density of primes. From the Prime Number Theorem (see J. Derbyshire,”Prime Obsession”) we know that the fraction of primes in the range x_1<x<x_2 is given approximately by-

\[ Li(x_2, x_1) = \int_{\sigma(x_1)}^{\sigma(x_2)} \frac{dt}{\ln(t)} \]

provided that x_1>1.451369234… The lower limit is introduced because of the minus infinity occurring for Li(1). This logarithmic integral criterion produces results very close to the
actual number of primes while the older Prime Number Theorem of Gauss which states that the number is approximately \( x/\ln(x) \) lies somewhat below the actual value. Let us test the theorems out for the range \( x_1=10,000 \) to \( x_2=10,100 \). We find-

\[
\text{Li}(10,100)-\text{Li}(10,000)=10.851492.. \quad \text{and} \quad 10,100/\ln(10,000)-10,000/\ln(10,000)=9.673943..
\]

, while the present search method produces exactly 11 primes which are-

10007, 10009, 100037, 10039, 10061, 10067, 10069, 10079, 10091, 10093, and 10099.

Plotting \( \text{Li}(x, 1.451369) \) versus the value \( x \) of the \( n \)th prime we get the following graph-

Agreement is seen to be quite good. The number of primes can be gotten from the above graphing approach. However, to save time, we just used the built-in information on the first 10,000 primes in our MAPLE program. We have, for example, that the 6000nds prime equals 59,359. This fits nicely along the above curve. That the number of primes are infinite follows from the fact that \( \text{Li}(\infty) = \infty \).

A semi-prime is defined as a number \( N \) which equals the product of two primes \( p \) and \( q \). To factor such a number for large \( p \) and \( q \) is a time consuming task and accounts for the fact that such numbers are the basis of public key cryptography. In terms of \( f(N) \) such numbers have –

\[
f(N)=f(pq)=\frac{(p+q)}{N} = \frac{p^2 + N}{pN} \approx \frac{2}{\sqrt{N}}
\]

We are assuming that \( p<\sqrt{N}<q \). Large semi-primes will have values of \( f(N) \) near zero. Let us look at one of these semi-primes, namely \( N=2479 \). This number has \( f(2479)=0.0419524.. \) and the approximation \( 0.040169… \) Solving the equation for \( p \), we have-
Thus 2479 factors into \( p=37 \) and \( q=N/p=67 \). To get this result required us to first find \( f(N) \). The factoring process is thus limited to those semi-primes where \( f(N) \) can be readily found. We can also work out \( F(N) \) for this semi-prime. It yields \( F(2479)=0.009708648687.. \) and so appears to be a poor indicator for a semi-prime since it lies a long distance from unity. However, one needs to point out that \( F(N) \) for \( N=2478 \) and \( N=2480 \) both are an order of magnitude smaller than this value. For the larger semi-prime \( N=277847951 \) we find \( f(N)=0.0001377731952.. \) and the approximation 0.00011998… Plugging into the above formula produces-

\[
p = 9733 \text{ from which follows that } q=28547
\]

Note that-

\[
9733<\sqrt{N}=16668.77178..<28547
\]

When the approximation \( 2/\sqrt{N} \) is close to the actual value of \( f(N) \) then one can be fairly sure that \( N \) is a semi-prime. If \( f(N) \) is much greater than this approximation, the number will typically be composed of the product of three or more primes. The number \( N=134569547 \) has \( f=0.1533486027 \) while \( 2/\sqrt{N} \) has the much lower value of \( 0.0001724076771 \). It therefore cannot be a semi-prime. Indeed it factors into the three prime product \( (7)(109)(176369) \).

Once \( \sigma(x) \) is known, the number fraction \( f(x) \) can always be found and visa versa. We have-

\[
\sigma(N)=1+N+Nf(N)
\]

So if \( N=3^6=729 \), we get \( f(729)=364/243 \) and \( \sigma(729)=1093 \).

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