GEOMETRIC SERIES APPLIED TO FUNCTIONS OF COMPLEX VARIABLES

The infinite series-
\[ S[f(z), \infty] = 1 + f(z) + f(z)^2 + \ldots = \sum_{k=0}^{\infty} \{f(z)\}^k \]

is the geometric series for the complex variable function-
\[ f(z) = u(x,y) + iv(x,y) \]

It will converge provided that –
\[ |f(z)| = \sqrt{u(x,y)^2 + v(x,y)^2} < 1 \]

and under that condition yields-
\[ S[f(z), \infty] = \frac{1}{1 - f(z)} = \frac{(1 - u) + iv}{(1 - u)^2 + v^2} \]

So that \( S[(1/2)+i(1/3), \infty] = (18 + 12i)/13. \)

We want in this article examine this series and its truncated form in more detail.

Consider first the complex function \( f(z) = \exp(iz) = \cos(z) + i\sin(z) \) with \( |f(z)| < 1 \). Here the infinite sum equals-
\[ \sum_{k=0}^{\infty} \exp(ikz) = \frac{1}{1 - \exp(iz)} \]

Expanding using \( z = x + iy \) we find-
\[ \sum_{k=0}^{\infty} \exp(-ky) [\cos(kx) + i\sin(kx)] = \frac{[1 - \exp(-y)\cos(x)] + i[\exp(-y)\sin(x)]}{1 - 2\exp(-y)\cos(x) + \exp(-2y)} \]

Taking the real part of this identity and setting \( x = \pi/2 \) and \( y = \pi/4 \) produces-
\[
\sum_{k=0}^{\infty} \exp(-\pi k/4) \cos(\pi k/2) = \frac{1}{1 + \exp(-\pi/2)} = 0.8278971...
\]

Take next the complex variable function \( f(z) = 1/z \). That is-

\[
f(z) = \frac{x - iy}{x^2 + y^2}
\]

Here-

\[
\sum_{k=0}^{\infty} \frac{1}{z^n} = \frac{z}{z - 1} = \frac{(-x + x^2 + y^2) - iy}{(1 - x)^2 + y^2}
\]

So, when \( x = y = 1 \) we have-

\[
\sum_{k=0}^{\infty} \frac{1}{(1 + i)^k} = (1 - i)
\]

Again it must be remembered that \(|f(z)|<1\). This means, for instance, that the above equality would fail if we set \( x = 0 \) and leave \( y = 1 \).

If we let \( f(z) = 1/g(z) \) and \(|g(z)|>1\), then we also have-

\[
\sum_{k=0}^{\infty} \frac{1}{g(z)^k} = \frac{g(z)}{g(z) - 1}
\]

A special case for this equality when \( g(z) = \cosh(z) \) is-

\[
\sum_{k=0}^{\infty} \{\sec h(z)\}^k = \frac{\cosh(z)}{\cosh(z) - 1} = \frac{1}{1 - \sec h(z)}
\]

We next look at a truncated geometric series where the lower limit on the sum remains at \( k = 0 \) but the upper limit is set at the finite value \( k = n \). This produces the equality-

\[
\sum_{k=0}^{n} f(z)^k = \frac{1}{1 - f(z)} - \sum_{k=n+1}^{\infty} f(z)^k \quad \text{when} \quad |f(z)| < 1
\]

On expanding the left series we find-
\[(1 - f(z)[1 + f(z) + f(z)^2 + \ldots + f(z)^n] = 1 - [1 - f(z)] \sum_{k=n+1}^{\infty} f(z)^k\]

But on multiplying the left part of this equality, we find-

\[1 - f(z)^{n+1} = 1 - [1 - f(z)] \sum_{k=n+1}^{\infty} f(z)^k\]

Hence we arrive at the important result that-

\[\sum_{k=0}^{n} f(z)^k = \frac{1 - f(z)^{n+1}}{1 - f(z)}\]

From it we see at once that-

\[\sum_{k=0}^{99} [0.99]^k = 100(1 - 0.99^{100}) = 63.39676587\ldots\]

and-

\[\sum_{k=0}^{29} \left[ \frac{1}{1+i} \right]^k = (1-i) \{1 - \frac{1}{(1+i)^{30}}\} = .9999694824 - 1.000030518i\]

Finally let us look at the double truncated geometric series-

\[\sum_{k=a}^{b} f(z)^k = \frac{1}{1 - f(z)} - \sum_{k=0}^{a-1} f(z)^k - \sum_{k=b+1}^{\infty} f(z)^k = \frac{f(z)^a - f(z)^{b+1}}{1 - f(z)}\]

Thus we have-

\[\sum_{k=10}^{20} \frac{1}{2^k} = \frac{(1/2)^{10} - (1/2)^{21}}{(1/2)} = 0.001952171326\]

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Happy Easter
and April Fools Day