DERIVATION OF THE BASIC LAWS OF GEOMETRIC OPTICS

It is well known that a light ray reflecting off of a surface has its angle of reflection equal to its angle of incidence and that if this ray passes from one medium to another that the interface condition is that the product of the index of refraction times the sine to the interface normal remains constant in both media. One finds numerous discussions on UTube concerning the derivation of these laws. Unfortunately these are mostly presented in a difficult to understand heavily accented manner and often much too long for what is being conveyed. We remedy this situation here by re-deriving the basic laws of geometrical optics starting with the Fermat Principle that a light rays will always choose a path which minimizes(actually extremizes) the transit time in going between two points.

We begin our derivations and discussions with the following simple diagram-

DIAGRAM SHOWING INCIDENT, REFLECTED, AND REFRACTED LIGHT RAY PLUS SURFACE NORMAL
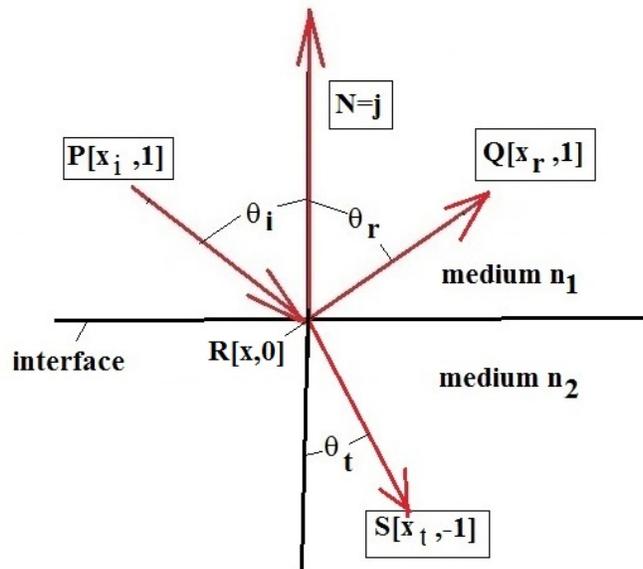
It indicates an incoming light ray emanating at point $P[x_i,1]$, hitting a flat surface at $R[x,0]$, and then being partially reflected toward point $Q[x_r,1]$ and partially refracted going toward point $S[x_r,-1]$ in the second medium. The three angles associated with these rays relative to the surface normal N are given as indicated.

LAW OF REFLECTION:

Here we start at point $P[x_i,y]$, hit the interface at point $R[x,0]$, and then send its reflected portion to $Q[x_r1]$. The time it takes to go from point P to Q along this two part path is just-

$$T = \frac{n_1}{c}\left\{\sqrt{(x_i - x)^2 + 1} + \sqrt{(x - x_r)^2 + 1}\right\}$$

Here $c = 3 \times 10^{10}$ cm/sec is the speed of light in vacuum and n_1 the index of refraction in the first medium. Typical values for n are 1 for vacuum, 1.00029 for air, 1.33 for water, 1.52 for crownt glass, and 2.4 for diamond.

Taking the derivative of T with respect to x and setting the result to zero produces the Fermat condition -

$$\frac{(x - x_i)}{\sqrt{(x - x_i)^2 + 1}} = \frac{(x_r - x)}{\sqrt{(x_r - x)^2 + 1}}$$

This is equivalent to saying $\sin(\theta_i) = \sin(\theta_r)$, where the angles are measured relative to the surface normal N=j at the reflection point R[x,0]. Thus we have the Law of Reflection-

$$\theta_i = \theta_r$$

When this result is used for ray-tracing through complicated optical systems such as in certain solar mirror arrays, one usually prefers the vector form of this result. This vector form can be derived as follows. Start with the vector representations of the incoming and reflected ray-

$$V_i = [ai - bj]/\sqrt{a^2 + b^2} \quad and \quad V_r = [ai + bj]/\sqrt{a^2 + b^2}$$

Next take the difference to get-

$$V_r - V_i = 2bj/\sqrt{a^2 + b^2}$$

But N=j and $b/\mathrm{sqrt}(a^2+b^2) = \cos(\theta) = -(N \cdot Vi)$. Hence the resultant vector form of the reflection law becomes-

$$V_r = V_i - 2(V_i \cdot N)N$$

A rather involved application of this formula can be found in one of the technical articles I published back in 1980 in Solar Energy 25, pp221-223.

CORNER REFLECTORS:

Another important application of the refection law applies to corner reflectors. In its simplest form a corner reflector consists of two perfectly reflecting mirrors mounted at right angles to each other as shown-

BEHAVIOUR OF A LIGHT RAY AT A CORNER REFLECTOR

incoming ray
$V_{in} = [-ai+bj]/\sqrt{a^2+b^2}$

outgoing ray
$V_{out} = [\ ai-bj]/\sqrt{a^2+b^2}$

$(V_{in} \cdot V_{out}) = -1$
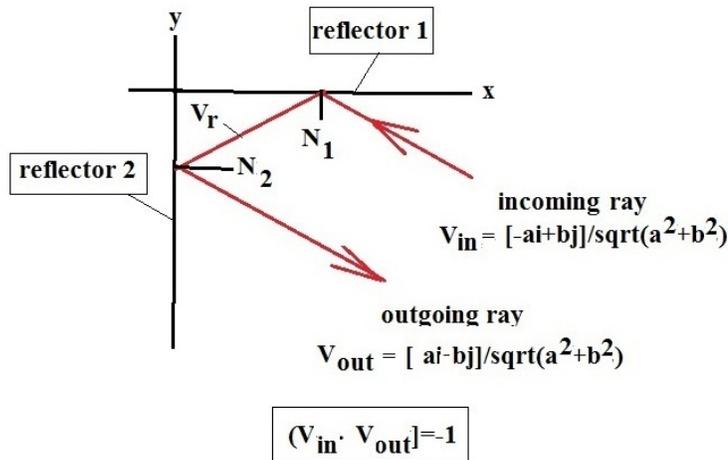
equivalent to a 180 deg phase shift

An incoming ray hits the reflecting surface 1 with an incident angle θ_i relative to thye normal N_1. On reflection $\theta_i = \theta_r$. A second reflection at normal N_2 has incident and reflected angles equal to $(\pi/2 - \theta_i)$. Hence adding up all four angles we get $2\theta_i + (\pi - 2\theta_i) = \pi$. Thus vectors Vin and Vout are 180 deg out of phase with each other meaning what goes in is reflected back in the same direction regardless of the corner reflector orientation. Billard players will be very familiar with this effect. Adding a third reflector at right angles to the first two does not change the results. It is this property of corner reflectors which makes them useful in measuring distances using laser pulses. Indeed , during one of the earliest trips to the moon , astronauts left a 3D corner reflector lying on the moon surface allowing observers from earth to make very accurate determinations of the earth-moon distance.

LAW OF REFRACTION:

To generate the Law of Refraction also known as Snell's Law, we consider points $P[x_i,1]$,$R[x,1]$ and $S[x_t,1]$ in the above diagram. This time the time for the light ray to go from P to S is given by-

$$T = \frac{n_1}{c}\sqrt{(x-x_i)^2+1} + \frac{n_2}{c}\sqrt{(x_r-x)^2+1}$$

On differentiating once with respect to x and setting the result to zero, produces the time minimum condition-

$$n_1 \frac{(x - x_i)}{\sqrt{(x - x_i)^2 + 1}} = n_2 \frac{(x_t - x)}{\sqrt{(x - x_{t_r})^2 + 1}}$$

From the diagram we recognize the quotients in this expression to be just the sine of θ_i and θ_t. Hence we have the Law of Refraction, also known as Snell's law, -

$$n_1 \sin(\theta_i) = n_2 \sin(\theta_t)$$

This important law forms the basis for most lens calculations employing ray tracing. Notice that when $n_1 < n_2$ a condition may exist where a light ray coming from S will be entirely internally reflected at the interface. This condition is known a total internal reflection and plays a major role in lossless light pulse transmission through optical fibers. If you are lying at the bottom of a swimming pool and looking up you will only be able to see the air side within a circle forming a cone with your eyes at the vertex. This cone has the half-angle -

$$\theta_{half\ cone} = \arcsin(\frac{1}{1.33}) = 48.75 \deg$$

SIMPLE EXAMPLE OF RAY TRACING:

To demonstrate how the Refraction Law is used in ray tracing, we consider the following optical set-up-

DEFLECTION ANGLE USING RAY TRACING
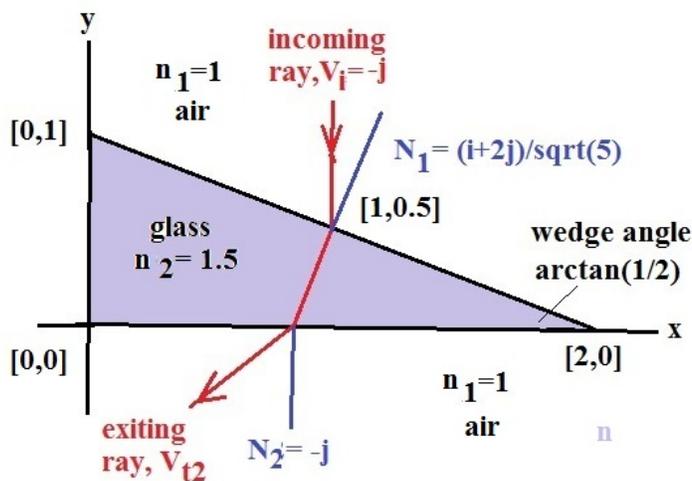
and try to find the deflection angle θ_{t2} between the exiting ray V_{t2} and the second normal N_2. We have here a glass wedge of vertex angle $\theta_w=\arctan(1/2)$ and index of refraction $n_2=1.5$. A light ray $V_i=-j$ comes in parallel to the y axis and hits the top of the wedge at point [1,0.5].The surface normal at the top of the wedge is determined by a gradient operation and just equals-

$$N_1 = (i + 2j)/\sqrt{5}$$

, when adjusted to have unit length. The angle of incidence with respect to this normal is just-

$$\theta_{i1} = \arccos(j \cdot N_1) = \arccos(2/\sqrt{5}) = 26.56505 \deg$$

By Snell's Law this yields the refraction angle θ_{t1} of-

$$\theta_{t1} = \arcsin(\frac{2}{3\sqrt{5}}) = 17.34606 \deg$$

The refracted ray now moves in a straight line along the θ_{t1} direction until it hits the bottom surface. There it is again refracted exiting the bottom of the wedge at angle θ_{t2} relative to the bottom normal $N_2=-j$. A little geometry involving the oblique triangles existing within the wedge produces an incident angle at the lower surface of -

$$\theta_{i2} = \theta_w - \theta_{t1}$$

From this last result it follows that the deflection angle is-

$$\theta_{t2} = \arcsin\left\{\frac{3}{2}\sin[\arctan(1/2) - \arcsin(2/3\sqrt{5})]\right\} = 13.904 \deg$$

Note that this ray will eventually cross the y axis. It gives an indication of how lenses work to produce a focal point.

REFLECTION COEFFICIENT AT AN INTERFACE:

A remaining formula needed in geometrical optics calculations is one relating the reflected ray intensity compared to the incident intensity. For this calculation it becomes necessary to consider light as a transverse wave with tine dependent electric and magnetic fields at right angles to the propagation direction. The first calculation using such a model was carried out by the French physicist and engineer Augustin-Jean Fresnel (1788-1827). Although better known to the public for his discovery of Fresnel lenses for

light houses, his four equations governing light transmission and reflection an interface ,motivated by earlier work of Young and predating Maxwell by some forty years, was truly a remarkable achievement. Let us show how to derive one of these equations based upon a polarized light wave where its electric field E lies perpendicular to the page in the above diagram. Under these conditions the magnetic field B lies in the plane of the paper. The boundary conditions at the interface are that-

$$E_i + E_r = E_t \quad \text{and} \quad B_i \cos(\theta_i) - B_r \cos(\theta_r) = B_t \cos(\theta_t)$$

Also we have the electromagnet waves relation-

$$B = nE/c$$

applicable for incident, reflected and transmitted waves. Eliminating the Bs and E_t from these equations yields-

$$n_1\{E_i \cos(\theta_i) - E_r \cos(\theta_r)\} = n_2\{[E_i + E_r]\}$$

Solving for the ratio E_r/E_i we get-

$$\frac{E_r}{E_i} = \left\{ \frac{n_1 \cos(\theta_i - n_2 \cos(\theta_t)}{n_1 \cos(\theta_r) + n_2 \cos(\theta_t)} \right\}$$

One can simplify this result further by using the reflection law $\theta_i = \theta_r$ and the Snell's Refraction Law that –

$$\theta_t = \arcsin\{\frac{n_1}{n_2} \sin(\theta_i)\}$$

Now we know that the ratio of reflected light intensity to the incident intensity can be considered as the Reflection Coefficient R and is represented by the square of Er/Ei . A case of special interest occurs when the incident light hits the interface along the normal direction. In that case $\theta_i = \theta_r = \theta_t$ and the reflection coefficient reduces to the simple form-

$$R = \left| \frac{(n_2 - n_1)}{(n_2 + n_1)} \right|^2$$

For an air-glass interface we find approximately $R = (0.5/2.5)^2 = 0.04$. That is only about 4% of the incoming incident orthogonal ray is reflected. The rest is transmitted. That is the reason that one looses very little intensity of the incoming light when looking through a pane of glass.

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