VARIATIONS ON THE GEOMETRIC SERIES

In one of the above notes from 2004 we discuss the basic geometric series and some of its properties when the number of terms in the series is infinite. We want here to extend this discussion to geometric series with finite terms and to look at some of its variations. Start with the series-

$$S(r, N) = \sum_{n=1}^{N} r^n = r + r^2 + r^3 + \ldots + r^N$$

If one subtracts this series from $rS(r, N)$, one finds at once that the series sums to-

$$S(r, N) = \frac{r(1 - r^N)}{1 - r}$$

As long as $N$ remains finite and $r$ is unequal to unity, this series produces a finite value. When the value of $N$ becomes infinite, then the absolute value of $r$ must remain less than one for convergence of the series. Notice that one can replace $r$ by a function $f(z)$ where $z=x+iy$ is a complex variable and the results should still hold. Thus we have-

$$\sum_{n=1}^{N} f(z)^n = \frac{f(z)[1 - f(z)^N]}{1 - f(z)}$$

If now $f(z)=\exp(iz)=\exp(-y)[\cos(x) + isin(x)],$ we find-

$$\sum_{n=1}^{\infty} \exp(inz) = \frac{1}{\exp(-iz) - 1}$$

Taking the real part of both sides yields the rather complicated result-

$$\sum_{n=1}^{\infty} \exp(-ny) \cos(nx) = \frac{[e^y \cos(x) - 1]}{[e^{2y} - 2e^y \cos(x) + 1]}$$

It is unlikely that this result will be found in any handbook or that a computer will spit out the analytic form shown on the right. That it is correct may be verified by choosing $y=1$ and $x=\pi/4$. In that case both sides of the equation yield the same result of $0.20289351\ldots$. Taking another function $f=\sin(x)/x$, we find-
\[
\sum_{n=1}^{\infty} \left[ \frac{\sin(x)}{x} \right]^n = \frac{\sin(x)}{x - \sin(x)}
\]

A third function considered is \( f = \exp(x) \). It produces the identity-

\[
\sum_{n=1}^{N} \exp(-xn) = \frac{[1 - \exp(-xN)]}{[\exp(x) - 1]}
\]

For \( x = 1 \), it produces-

\[
\frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \ldots + \frac{1}{e^N} = \frac{[1 - e^{-N}]}{[e - 1]}
\]

The following graph shows how the sum approaches a constant value as \( N \) gets large-
Next, let us go back the above expression $S(r,N)$ and differentiate it a few times with respect to $r$. This produces the results-

$$\sum_{n=1}^{N} nr^{n-1} = \frac{1}{(1-r)^2} \left\{ 1 - (N+1)r^N + Nr^{N+1} \right\}$$

and-

$$\sum_{n=1}^{N} n(n-1)r^{n-2} = \frac{1}{(1-r)^3} \left\{ 2 - N(N-1)r^{N+1} + 2r^N (N^2 - 1) - r^{N-1}N(N-1) \right\}$$

If $|r|<1$, one can surmise from these results that-

$$\sum_{n=1}^{\infty} n(n-1)(n-2)\ldots(n-k)r^{n-k-1} = \frac{(k+1)!}{(1-r)^{k+2}}$$

Thus at $k=3$ and $r=1/2$, we find the identity–

$$\sum_{n=1}^{\infty} \frac{(n^2 + n)(n^2 - 5n + 6)}{2^n} = 48$$

The following sum taken to $N=100$ yields the value shown-

$$\sum_{n=1}^{100} n2^{n-1} = 99 \cdot 2^{100} + 1$$

$$= 125497409422594710748173617332225$$

Notice that this type of sum will blow up as $N$ approaches infinity since $r>1$.

Finally let us look at a sum for which $f=z^n$ and where we replace $z$ by its polar form $r \exp(i\theta)$. This produces-

$$\sum_{n=1}^{N} r^n [\cos(n\theta) + i \sin(n\theta)] = \frac{re^{i\theta} \left[ 1 - r^N e^{iN\theta} \right]}{1-re^{i\theta}}$$

Taking the real and imaginary parts and letting $N \rightarrow \infty$, we find the results-
\[ \sum_{n=1}^{\infty} r^n \cos(n \theta) = \frac{-r[r - \cos(\theta)]}{1 - 2r \cos(\theta) + r^2} \]

and-

\[ \sum_{n=1}^{\infty} r^n \sin(n \theta) = \frac{r \sin(\theta)}{1 - 2r \cos(\theta) + r^2} \]

These results are only valid when \( r < 1 \). You may recall that this kind of sum was encountered when working out the Poisson Integral for the Circle.

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