MANIPULATIONS WITH THE GEOMETRIC SERIES

One of the simplest and at the same time useful infinite series encountered in analysis is the geometric series -

\[ G(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} \quad \text{or its equivalent form} \quad \sum_{n=1}^{\infty} x^n = \frac{x}{1 - x} \]

In both cases it is necessary that the absolute value of x remain less than one for the series to converge to the value on the right of the equality sign. The G(x) series was already well known to Huygens, Leibnitz, and Newton and can be easily established by subtracting G(x) from xG(x). What is often not recognized by students and others is that this series can lead to an infinite number of alternative equalities via the substitution \( x \rightarrow f(x) \) and the subsequent use of differentiation and integration. All that is necessary is to keep in mind that \( |f(x)| < 1 \).

Let's demonstrate these points by first looking at \( x \rightarrow \exp(-kx) \) with \( kx > 0 \). This leads at once to

\[ \exp( -kx ) = \exp(-k) \sum_{n=1}^{\infty} \frac{(-kx)^n}{n!} \]

On setting \( x = 1 \) and integrating this last expression over \( 0 < k < \infty \), one finds

\[ \sum_{n=1}^{\infty} \frac{\exp(-kn)}{n} = -k \int_{1}^{\infty} \frac{dk}{\exp(k) - 1} = -\ln[1 - \exp(-k)] \]

Thus next time someone asks you what is the value of the infinite series defined by the function \( \exp(-n)/n \) summed from \( n = 1 \) to infinity, you can give them the answer 0.458675145,.. as is found at once by evaluating \( -\ln[1-\exp(-1)] \) with your hand calculator. Note that in the above expression one recovers the harmonic series when \( k \rightarrow 0 \) and thus sees it is divergent since \( -\ln(0) \) is infinite. On integrating the above equality one more time with respect to \( k \), and this time using the range(0, k), we obtain

\[ \sum_{n=1}^{\infty} \frac{\exp(-kn)}{n^2} = \frac{\pi^2}{6} + \int_{0}^{k} \ln[1 - \exp(-k)]dk \]

where the integral on the right is expressible in terms of the dilogarithm function (see Abromowitz and Stegun, "Handbook of Mathematical Functions"). In obtaining this last result, we made use of the famous Euler result that the sum of the square of the reciprocals of all the
integers equals $\pi^2/6=1.644934067$. Letting $k\to\infty$, we find that -

$$
\int_0^1 \frac{\ln(u)}{(1-u)} \, du = -\frac{\pi^2}{6}
$$

I remember encountering this singular integral back in my calculus class many years ago and recall that I was unable to work out its value analytically at that time although it was clearly correct as a numerical evaluation showed. Integrals of this type also arise in connection with the Debye theory of specific heats. Differentiating the exponential sum given above will also lead to numerous additional identities. One such identity (which I leave for the reader to derive) is-

$$
\sum_{n=1}^{\infty} n^6 \exp(-nx) = \frac{\sinh(x)[\cosh(x)^2 + 28\cosh(x) + 61]}{2[\cosh(x) - 1]^3}
$$

For $x=1$ it has the numerical value $719.99693557164...$. The fact that this value lies close to 720 should not be surprising in view of the fact that the gamma function of 7 equals $6!$.

As a second substitution consider $x\to x^k$ with $k>0$ and perform an integration over the range $(0,x)$. This produces the identity-

$$
\sum_{n=0}^{\infty} \frac{x^{nk+1}}{nk + 1} = \int_0^x \frac{dx}{(x^{-k} - 1)}
$$

The integral is here expressible in terms of the LerchPhi function (see MAPLE). An interesting special case occurs for $k=1$ and $x=1/2$. It reads-

$$
\frac{1}{2^1(1)} + \frac{1}{2^2(2)} + \frac{1}{2^3(3)} + \frac{1}{2^4(4)} + \ldots = 0.5 + \int_{0}^{1/2} \frac{x}{(1-x)} \, dx = \ln(2)
$$

For our final example, consider the substitution $x\to-x$ and then summing $G(x)+G(-x)$. This produces the well known logarithmic result-

$$
\frac{1}{2} \ln \left[ \frac{1+x}{1-x} \right] = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)}
$$

which can be used to rapidly calculate the logarithms of numbers such as 2. Setting $x=-1/3$, one finds-
\[
\ln(2) = \frac{2}{3} \sum_{n=0}^{\infty} \frac{1}{3^{2n} (2n + 1)} = \frac{2}{3} \left[ \frac{1}{3^0} + \frac{1}{3^2} + \frac{1}{3^4} \ldots \right] = 0.69314718..
\]

This last result comes in very handy when discussing things such as compound interest. It shows, for example, that one’s money will double in ten years at a 6.93% interest (provided of course that taxes and inflation are neglected).

Finally, we consider the truncated geometric series:

\[
S(x, N) = \sum_{n=0}^{N} x^n = 1 + x + x^2 + \ldots + x^N
\]

and drop the requirement that \(|x|<1\). Although this will produce infinite values for \(S(x,N)\) as \(N\) approaches infinity and \(x\) is greater than unity, it produces a finite value when \(N\) is finite. By looking at-

\[
S(x, N) - xS(x, N) = 1 - x^{N+1}
\]

one sees at once that-

\[
S(x, N) = \frac{(x^{N+1} - 1)}{(x - 1)}
\]

We thus have that-

\[
1 + 2 + 4 + 8 + 16 + \ldots + 2^{10} = 2^{11} - 1 = 2047 = 23 \times 89
\]

which happens to be a counterexample of the Mersenne prime conjecture that any number of the form \(2^p - 1\) will be a prime number provided that \(p\) is prime. Also it is seen on replacing \(x\) by \([\cosh(x)]^2\) that-

\[
\sum_{n=0}^{N} \cosh(x)^{2n} = \cosh(x)^{2N} \coth(x)^2 - \csc h(x)^2.
\]

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