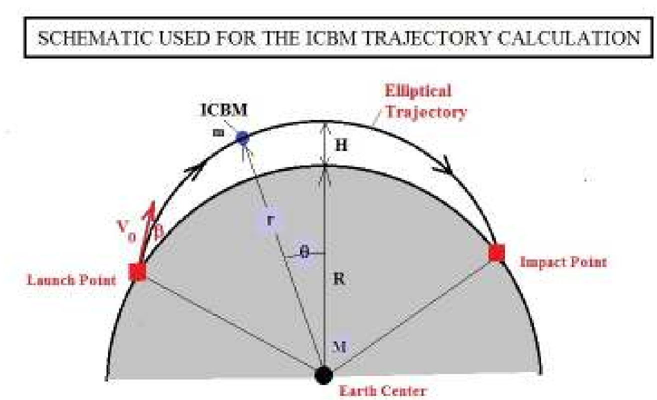
CALCULATING THE TRAJECTORY AND IMPACT TIME OF AN ICBM

During the peak of the cold war in the 1950s both the US and Russia installed thousands of nuclear tipped ICBMs at various points throughout their countries aimed at the opponents major cities and missile silos. It was a very dangerous game which came to be known as the Principle of Mutual Assured Destruction (MAD). The standoff lasted until the late 1980s when Russia realized that it could no longer compete economically, especially in view of the newly proposed star wars program. Today such a nuclear standoff may be shifting to the Middle East especially between Israel and Iran, although it seems much more likely that we will see an asymmetric event involving a suicide bomber from Pakistan, Somalia, or Yemen setting off a backpack sized nuclear bomb. It is also possible that in the future a MAD standoff may occur between the US and China.

Be that as it may, we want here to look at the characteristics of an ICBM as it moves from a launch point to the final impact point. We want to do this with a minimum of mathematics, yet be able to calculate such things as maximum height H reached above the earth's surface and the time τ and range L to impact.

Our starting point for this discussion is the following schematic showing a typical ICBM trajectory-



We know that an ICBM follows an elliptical path starting from its launch point until impact. This trajectory lies essentially in a plane defined by the launch and impact points and the earth center. Neglecting all air friction, we can apply the conservation of energy law and angular momentum conservation to get the two equalities-

$$\frac{1}{2}m({v_r}^2+{v_\theta}^2)-\frac{GMm}{r}=\frac{1}{2}m{V_0}^2-\frac{GMm}{R}$$

and-

$$rv_\theta=RV_0cos(\beta)$$

Here the v_r and v_θ are the velocity components in the radial and angular direction, m the ICBM's mass, M and R the earth's mass and radius, V_0 and β the launch speed and launch angle, and G the universal gravitational constant. On combining these two equations, after setting GM= gR^2, where g is the standard acceleration of gravity at the earth's surface, one finds-

$$(\frac{v_r}{V_0})^2=1+\frac{2gR^2}{{V_0}^2}\left(\frac{1}{r}-\frac{1}{R}\right)-\frac{[Rcos(\beta)]^2}{r^2}$$

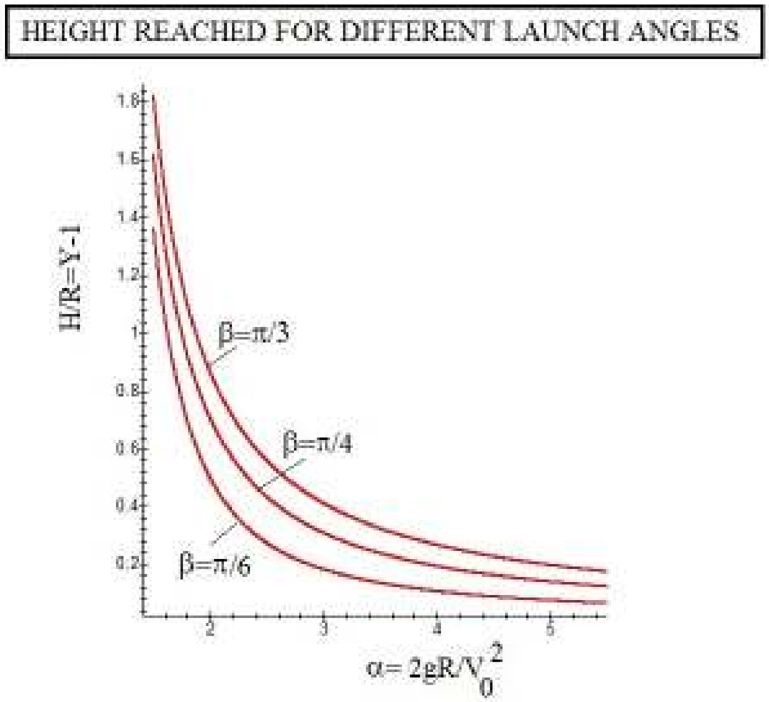
To find the highest point H above the earth's surface reached by the missile, we simply set v_r to zero. This produces the quadratic equation-

$$Y^2(1-\alpha)+\alpha Y-cos(\beta)^2=0$$

where Y=r/R and $\alpha=2gR/V_0^2$. Typically both Y and α will have values greater than one. An interesting obvious solution occurs for β=π/2 which corresponds to a vertical launch. It produces an infinity value for Y when α=1. This value represents the escape velocity V_0=sqrt(2gR). For the earth this number equals 11.2km/s. For a cos(β) different from zero, the quadratic equation produces one positive solution-

$$\frac{H}{R}=-1+[\frac{\alpha+\sqrt{\alpha^2-4(\alpha-1)(cos\beta)^2}}{2(\alpha-1)}]$$

We have evaluated this last result for an ICBM launched at angles of π/3, π/4, and π/6 radians. The plot showing H/R versus α follows-



One sees that the higher the launch velocity the higher the missile will rise and that this rise becomes infinite as one reaches α=1. The rise goes to zero as V_0 approaches zero. If we let cos(β)=0 and assume H/R<<1 so that α>>1, one recovers the classical result that $H=V_0^2/2g$.

We can use the above calculated radial velocity component to solve for the time it takes for the missile to reach its maximum height and multiply this result by two (due to problem symmetry) to get the time of impact τ. The calculation reads-

$$\tau = \frac{2R}{V_0}\int_{Y=1}^{1+\frac{H}{R}} \frac{YdY}{\sqrt{Y^2(1-\alpha)+\alpha Y - cos(\beta)^2}}$$

Let us evaluate this integral. For this purpose we choose the realistic values cos(β)=sqrt(3)/2 and α=4 for which H/R=sqrt(7)/6-1/3=0.1076. The launch speed and angle are V_0=5.593 km/s and β=30°, respectively. Using R=6378 km,

we find τ=991sec=16.5 min. This number is close to the twenty minute or so warning time the US had in case of a Russian missile attack.

The question remains what is the surface distance L between the launch and impact point in this situation. We know from the conservation of angular momentum that $v_\theta = RV_0\cos(\beta)/r$ and for H/R<<1 that r is well approximated by R. Thus we have the range-

$$L \approx \tau V_0 \cos(\beta) = 991 \times 5.593 \times 10^3 \times \mathrm{sqrt}(3)/2 = 4800\text{km}$$

for the above case. This number is somewhat shorter than the 7525km distance between Moscow and New York. To increase the range one needs to simply increase V_0 and β. One can put the missile into a permanent circular orbit by having V_0=sqrt(gR)=7.9 km/s and β approach zero. The 7.9 km/sec will be familiar to many of the engineers among you as the 25,900 ft/sec speed required for a low earth orbit.

As a final remark, it must be remembered that the earth rotates so an ICBM is chasing a moving target. The adjustment is however quite easy to make and requires a longitude correction of (τ/1day)2πRcos(LAT) which can be built into the launch code. In the case of Moscow (LAT=55.75N, LONG=37.62E) one aims for a point approximately 313 km east of Moscow at the start of a τ=20 minute missile trip.

This rotation of the earth also accounts for the fact that a spy satellite moving in a polar orbit can observe every point on the earth below it without needing to make any orbit corrections.

In real launches of ICBMs one always starts with a stationary vertical launch but then quickly brings the missile up to the desired speed V_0 and launch angle β during the first few minutes of the trip. For the majority of its trajectory the missile is in free flight. This makes possible an anti-missile defense although none exists at this time for long range and high speed ICBMs. The Scud missiles used by Iraq were relatively slow moving (Mach 5) and were thus partially neutralized by a Raytheon anti-missile system. You may recall at the dawn of the missile age that the British developed an effective defense against the V1s

using high speed fighter planes and destroying launch sites but were helpless against the mobile based and supersonic V2s.

An interesting side light concerning ICBMs is the following 1945 quote by President Roosevelt's science advisor Vannevar Bush-

"A workable ICBM is an impossibility"

This is as bad a prediction as the 1933 statement by the Lord Rutherford that-

"Anyone who claims the nucleus to be a source of power is talking moonshine"