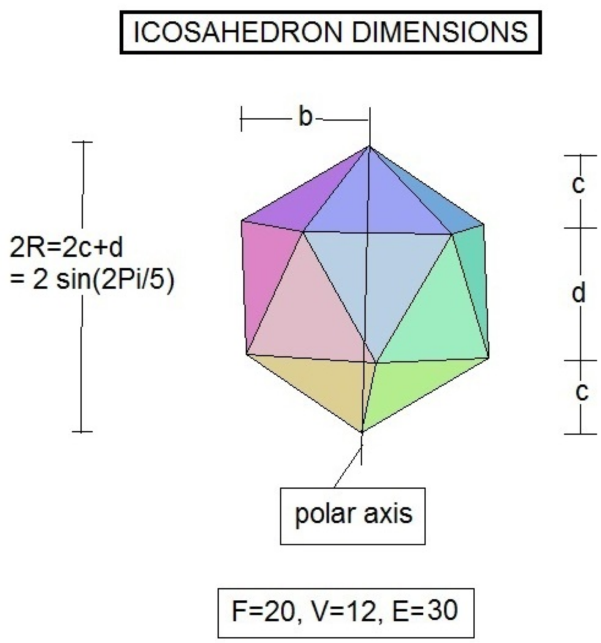
PROPERTIES AND CONSTRUCTION OF ICOSAHEDRA

The icosahedron is the most complicated of the five regular platonic solids. It consists of twenty equilateral triangle faces (F=20), a total of twelve vertices (V=12), and thirty edges (E=30). That is, it satisfies the Euler Formula that F+V-E=2. To construct this 3D figure one needs to first locate the coordinates of its vertices and thern can tile the structure with simple equilateral triangles. We will do this in the present article using a somewhat different than usual approach via cylindrical coordinates.

As a starting point we look at the following schematic of an icosahedron-



We draw a polar axis through the figure passing through the top and bottom vertex point. Also we notice that the top and bottom of the figure consists of a pentagonal pyramid of height c. the height of the girth of the figure is taken as d so that the circumscribing sphere has diameter D=2R=2c+d. All twelve vertices of the icosahedron lie on this sphere.
Looking at the pentagon base of the upper cap, we take the length of each side of the pentagon to be one. This means that the distance from each of the pentagon vertex points to the polar axis is –

$$b = \frac{1}{2\sin(\pi/5)} = \sqrt{\frac{2}{5-\sqrt{5}}} = \sqrt{\frac{\varphi}{\sqrt{5}}} = 0.8506508...$$

Here φ=[1+sqrt(5)]/2=1.61803398.. is the well known Golden Ratio

The distance from the middle of one of the pentagon edges to the polar axis is-

$$a=\frac{1}{2\tan(\pi/5)}=\frac{1}{2\sqrt{5-2\sqrt{5}}}=\frac{1}{2}\sqrt{\frac{3+4\varphi}{5}}=0.68819096...$$

.Now the height c of the pentagonal pyramid follows from the Pythagorean Theorem as-

$$c=\sqrt{\frac{3}{4}-a^2}=\frac{1}{2}\sqrt{3-\frac{1}{5-2\sqrt{5}}}=0.5257311110...$$

One notices from the above schematic, the pentagonal pyramid of the lower cap is rotated about the polar axis by π/5 radians. This means that the girth height d will be a little less than sqrt(3)/2. It is given by the formula-

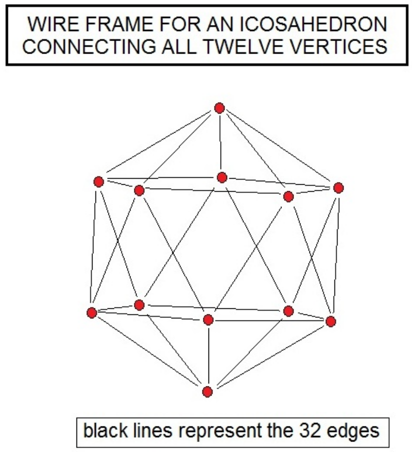
$$d=\sqrt{\frac{3}{4}-(b-a)^2}=0.8506508085...$$

Adding together the cap heights and the girth height then yields the radius R of the circumscribing sphere to be-

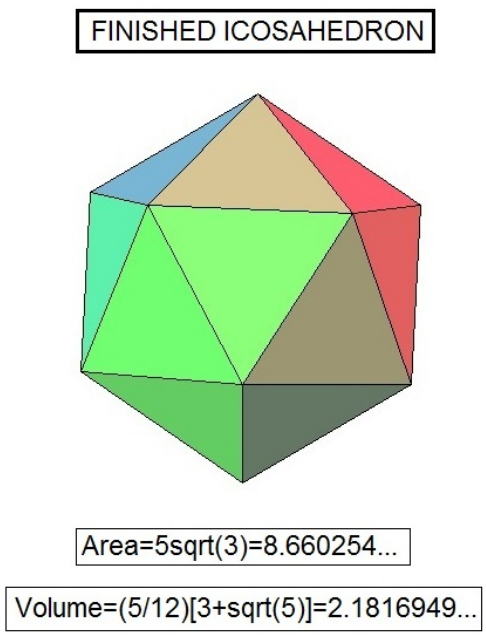
$$R=c+\frac{1}{2}\sqrt{\frac{3}{4}-(b-a)^2}=0.9510565...$$

An interesting fact about R is that it also equals the value of sin(2π/5) exactly.

We are now in a position to locate the position of all twelve vertex points. This is easiest to do by using cylindrical coordinates where the origin is located at the center of the icosahedron. This way the coordinates [r,θ,z] for the extreme upper and lower vertex points fall at [0,θ,±(c+d/2], respectively. The five vertices along the upper pentagon boundary are located at [b,2nπ/5,d/2]. For the lower pentagon cap the vertex points fall at [b,(2n+1)π/5,-d/2] due to the π/5 rotation about the z axis between the top and bottom caps. Here n=0,1,2,3, and 4. Plotting these 12 points and connecting them by straight lines we get the following picture-



On tiling this structure with equilateral triangles of side length one, we get the completed regular icosahedron with unique area and volume values-



The total area of the 20 faces is just twenty times the equilateral triangle area of sqrt(3)/4. It equals-

$$Area = 5\sqrt{3} = 8.6602540...$$

To get the volume we think this solid to be composed of 20 pyramids with the base of an equilateral triangle and height H given by-

$$H=\mathrm{sqrt}(R^2-1/3)=0.7557613129\ldots$$

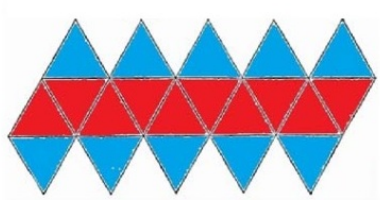
Since the volume of a pyramid just equals one third of its base area times the height, we get-

$$\text{Volume}=5H/\text{sqrt}(3)=(5/12)[3+\text{sqrt}(5)]=2.181694989\ldots$$

Note that one can express the above constants a, b ,c, d, and R in terms of functions of the golden ratio $\varphi=[1+\text{sqrt}(5)]/2$. This should not be surprising since when quantities contain sqrt(5), as these do , one can always use the substitution $2\varphi-1=\text{sqrt}(5)$.

Although the above detailed calculations have given one all the required information to construct icosahedra of any size, there are alternate ways to create these using very little mathematics. One of these procedures involves constructing a 2d net connecting twenty equilateral triangles and then folding. The first individual to come up with such a net was the brilliant German artist Albrecht Duerer, who in his 1525 book on perspective presented the following 2d pattern-

DUERER'S 1525 2D NET FOR CONSTRUCTING AN ICOSAHEDRON

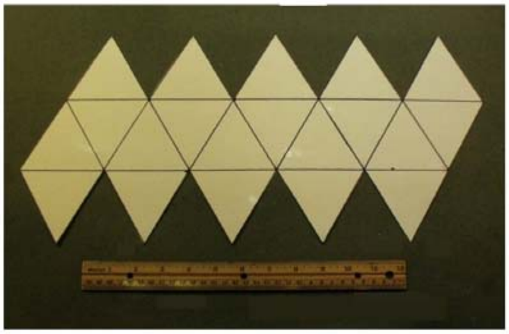


Faces(F=20), Edges (E=30), Vertices(V=12)

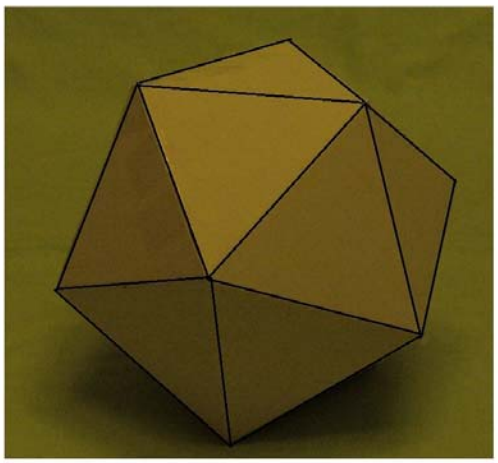
All faces are Equilateral Triangles

By folding along the edges the blue triangles construct the upper and lower cap of an icosahedron while the red triangles produce the enclosing girth. Duerer obviously had tremendous talent in visualizing three dimensional objects from 2d surfaces and visa-versa as reflected in his many paintings and etchings. The following are two configurations I worked out this morning using heavy construction paper. The first is the Duerer Net used and the second the resultant 3D structure-

2D DUERER NET FOR AN ICOSAHEDRON



CARDBOARD MODEL OF AN ICOSAHEDRON CONSTRUCTED FROM A DUERER 2D NET



Edges enhanced by black ink lines

Another approach for icosahedron construction is to cut out twenty equal sized equilateral triangles from a thin sheet of birch or other plywood and then connect them using small internal wooden blocks set at an interior angle of θ=138.189685*… degrees between neighboring surfaces. We constructed such a figure about a decade ago and have used it as a decorative 3D figure in my study since that time. It is quite rigid as expected for a near spherical shape and shows no signs of deterioration.

*-The angle follows from the formula-

$$\theta = \frac{180}{\pi}\arccos\{1-(\frac{2}{3})[b^2+(c+d)^2]\} = 138.19\,\mathrm{deg}$$

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November 11, 2018,
Gainesville, Florida
100th Anniversary of the End of WW1