Prior to the invention of calculus by Newton(1643-1727) and Leibnitz(1646-1716) in the late 16 hundreds, the standard way of determining areas under curves or volumes of solids was an incremental approach where things were broken up into sub-elements of decreasing but finite size. The master of this technique was the famous ancient Greek mathematician Archimedes of Syracuse(287-212 BC). He used such an approach to calculate the area under a parabola and the volume of a sphere. His approach was actually a precursor to latter day calculus. We want here to demonstrate this finite incremental method.

Let us begin with determining the area under a curve y=f(x). For the discussion we consider the curve to be the parabola y=x^2 in the range 0<x<1. First we break the area under the parabola into four trapezoids of width \( \Delta = 1/4 \) each and incremental area \( dA = (\Delta/2)(a+b) \), where \( a \) is the height of the left side of the sub-area and \( b \) the height of the right side. Adding the four sub-areas together yields the approximation:

\[
A_4 = \frac{1}{8 \cdot 16} [1 + 5 + 13 + 25] = 0.34375
\]

On breaking the area under the parabola into eight sub-trapezoids of width \( \Delta = 1/8 \) each we get the better area approximation:

\[
A_8 = \frac{1}{16 \cdot 64} [1 + 5 + 13 + 25 + 41 + 61 + 85 + 113] = 0.33593
\]

We could double again the number of sub-areas to 16 where \( \Delta = 1/16 \) and carry out the same type calculation. Fortunately this is not necessary since the terms in the square brackets can be generalized by noting that the jump in value from one term to the next goes as +4,+8,+12,+16,+20,+24,+28 etc Thus \( A_{16} \) becomes:

\[
A_{16} = \frac{1}{32 \cdot 256} [1 + 5 + 13 + 25 + 41 + 61 + 85 + 113 + 145 + 181 + 221 + 265 + 313 + 365 + 421 + 481] = 0.33398
\]

Clearly the incremental area goes to \( A_{\infty} = 1/3 \).
To calculate the area of a circle of radius $R$ we draw a series of concentric circles which produce annular rings of width $\Delta$ plus a small circle of radius $\Delta$ at the origin. Using the average radius of a ring as the distance from the origin to the center of a ring, we find the following area approximations:

$$A_2 \approx \frac{\pi R^2}{4} [1 + 2] = (3/4) \pi R^2$$

$$A_4 \approx \frac{\pi R^2}{16} [1 + 3 + 5 + 7] = \pi R^2$$

and-

$$A_8 \approx \frac{\pi R^2}{64} [1 + 3 + 5 + 7 + 9 + 11 + 13 + 15] = \pi R^2$$

So apparently we have already reached the limit –

$$A_\infty = \pi R^2$$

In summing up the odd numbers in the square brackets above we have made use of the identity-

$$\sum_{n=1}^{N} (2n - 1) = N^2 .$$

Although Archimedes used a polygon limiting procedure to find the area of a circle, he was perfectly capable of using the above much faster approach as shown in some of his later works involving the volume of a sphere.

Next consider Archimedes classical problem of the volume of a sphere. His approach was to take a stack $n=R/\Delta$ of horizontal discs of incremental volume $dV=\pi r^2 \Delta$, where the disc radius equals $r=\sqrt{R^2-z^2}$. He then added up the volumes from $z=0$ to $z=R$ and multiplied by two. This produces-

$$V_n = \frac{2\pi R^3}{n} \sum_{k=1}^{n} [1 - \left(\frac{z_k}{R}\right)^2]$$

Consider now placing $n=5$ discs of thickness $\Delta=0.2R$ and decreasing radius with increasing $z$. This yields-
\[ V_5 = 2\pi R^3 \left\{ 1 - \frac{1}{500} [1 + 9 + 25 + 49 + 81] \right\} = 1.34\pi R^3 \]

On looking at the terms in the square bracket we recognize the sum-

\[ S[5] = \sum_{n=1}^{5} (2n - 1)^2 = 165 \]

A generalization yields-

\[ S[m] = [11m + 8 - 4(m + 1)^2 (2 - m)] \]

So if we take ten equal thickness discs in half the sphere, we get the approximate sphere volume-

\[ V_{10} = 2\pi R^3 \left\{ 1 - \frac{1330}{4000} \right\} = 1.335\pi R^3 \]

For \( V_{20} \) we find-

\[ V_{20} = 2\pi R^3 \left\{ 1 - \frac{10.66 \cdot 0.625}{20} \right\} = 1.33375\pi R^3 \]

These results are all pointing toward a sphere volume of –

\[ V_\infty = 1.33333\pi R^3 = \frac{4}{3}\pi R^3 \]

Archimedes was so proud of this result that he had a sculpture of a sphere of radius \( R \) inside a cylinder of radius \( R \) and height \( 2R \) placed on his tombstone. He stated in one of his writings that the volume ratio of the two was exactly \( 2/3 \).

You will have noticed that in all of the above calculations one approaches an exact value as the increment \( \Delta \) is allowed to approach zero. That is exactly what one does in calculus, so one could say Archimedes was actually the father of calculus. What is trivial by today’s mathematical standards was certainly not back in 250 BC. One wonders where he was able to get all the writing material to carry out his work. Papyrus must have been very expensive even in Alexandria, Egypt where Archimedes spent part of his time. Perhaps he used wax tablets as the Romans did. Via calculus we get the above volume much faster. We simply have-

\[ V = 2\pi \int_{z=0}^{R} (R^2 - z^2)dz - 2\pi \left\{ R^2 z - \frac{z^3}{3} \right\}_{z=R}^{z=R} = (4/3)\pi R^3 \]
To calculate the length of curves by incremental methods is also straightforward. Take the case of a circle of radius R. Draw equally spaced radial lines from the origin to the circle so as to produce equal pie-slice shapes which can be approximated by isosceles triangles of sides R and vertex angle $\theta = \frac{2\pi}{n}$ rad. The short straight line base of each triangle is easy to calculate by geometry. It produces the total increment length around the circle of-

$$L[n] = 2R n \sin \left( \frac{\pi}{n} \right)$$

For $n=4$ it yields the not so good circumference approximation of $L[4]=4\sqrt{2}=5.656$ R. This just represents four times the side-length of an inscribed square. However when $n=8$ we get $L[8]=6.123$ R and when $n=16$ we have $L[16]=6.243$ R. Clearly by decreasing the straight line increments around the circle we get closer and closer to the exact circumference $L[\infty]=6.283$ R $= 2\pi R$.

June 6, 2017