We know that most functions, whether they represent a solution to a differential equation or not, can be expressed in terms of infinite series. This suggests that one should be able to evaluate many infinite series by simply recognizing what particular combination of such functions, at a given \( x \), constitutes the given summation in question. For example, the simple function \( f(x) = \exp(x) \) about \( x = 0 \) has the series expansion-

\[
\exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... = 2.718281828459045..
\]

so that the sum of the reciprocal of all factorials just sums to \( e \).

Likewise one finds for the hyperbolic Bessel function of the first kind, defined as,-

\[
I_\nu(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+\nu}}{n!(n+\nu)!}
\]

that-

\[
I_0(2) = \sum_{n=6}^{\infty} \frac{1}{(n!)^2} = 1 + \frac{1}{1!^2} + \frac{1}{2!^2} + \frac{1}{3!^2} + ... = 2.2795858302..
\]

Next we examine series summations obtainable via the hypergeometric series \( F(a,b,c,x) \). This function has been discussed in detail earlier and is defined as-

\[
F(a,b,c,x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \left[ \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \right] \frac{x^n}{n!} = 1 + \frac{ab}{c} x + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{x^2}{2!} + ...
\]

The ratio test readily shows for what values of \( a, b, c, \) and \( x \) this series will converge. In the limit of \( a = 1, b = c = 1 \), one obtains the standard geometric series-

\[
F(1,1,1,x) = 1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{provided} \quad |x| < 1
\]

For the case of \( a = 1/2, b = c = 1 \), and \( x = 1/2 \) we find that-
\[
\sum_{n=0}^{\infty} \frac{(2n+1)!}{(n+1)(n!)^2} 2^n = 1 + \frac{3}{8} + \frac{5}{32} + \frac{35}{512} + ... = 4(\sqrt{2} - 1) = 1.656854..
\]

and for \(F(1,2,3,1/2)\) one has-

\[
\sum_{n=0}^{\infty} \frac{2^{1-n}}{(n+2)} = 8 \ln(2) - 4 = 1.545177445..
\]

The complete elliptic integral of the first kind has the series representation-

\[
F(k) = \int_{x=0}^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin(x)^2}} = \pi \sum_{n=0}^{\infty} \left[ \frac{(2n)!}{4^n (n!)^2} \right]^2 k^{2n}
\]

From this last result, we have-

\[
\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} 4^{3n} = 1 + \frac{1}{16} + \frac{9}{1024} + \frac{25}{16384} + \frac{2F(0.5)}{\pi} = 1.073182007..
\]

Next look at some series involving hyperbolic and normal trigonometric functions. In particular, one finds that-

\[
\sinh(1) + \sin(1) = \sum_{n=0}^{\infty} \frac{1}{(4n+1)!} = 1.008336098..
\]

plus-

\[
[1 - \sin(1) \exp(-1)] = 1 - \frac{1}{3} + \frac{1}{30} - \frac{1}{90} + \frac{1}{630} - \frac{1}{22680} + ... = 0.6904401243..
\]

and-

\[
\sum_{n=0}^{\infty} (-1)^n a^{2n+1} = \int_{t=0}^{\infty} \sin(at) \exp(-t) dt = \frac{a}{a^2 + 1}
\]
Consider next the infinite series-

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} + \ldots = \frac{\sqrt{\pi}}{2}\text{erf}(1) = 0.7468241330\ldots
\]

From the above examples, it is seen that many infinite series may be summed if the values of certain transcendental functions are known at given points x. Very often the values of such functions are also obtained by summing an infinite series, but one which converges more rapidly. For example, to calculate erf(1) one can use an identity involving the confluent hypergeometric function \( M(a,c,x) \) to yield-

\[
erf(1) = \frac{2}{e\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{4^n n!}{(2n+1)!} = 0.8427007929\ldots
\]

Eliminating \( 2\text{erf}(1)/\sqrt{\pi} \) between the last two equations yields the quotient-

\[
\exp(1) = \frac{1 + \frac{4(1)!}{3!} + \frac{4^2(2)!}{5!} + \frac{4^3(3)!}{7!} + \ldots}{1 - \frac{1}{1!(3)} + \frac{1}{2!(5)} - \frac{1}{3!(7)} + \ldots}
\]

which gives a 35 place accurate result for \( e \) when taking the first 30 terms in both numerator and denominator.

To find the value of the hyperbolic Bessel function \( I_0(2) \) one can numerically integrate the integral representation-

\[
I_0(2) = \frac{1}{\pi} \int_{\pi/2}^{\pi} \cosh[2\cos(t)] dt
\]

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