PROPERTIES OF INTEGER SPIRALS

Several decades ago while teaching an undergraduate course in complex variables here at the University of Florida we encountered the point function F(z,n)=$(1+i)^n$, where the ns are the positive integers 1,2,3,4,5, . The first few of these numbers, expressed in polar coordinates within the z plane, read-

n	F(Z,n)=r exp(iθ)
1	sqrt(2)exp(iπ/4
2	2exp(iπ/2)
3	2sqrt(2)exp(3iπ/4)
4	4sxp(iπ)
5	4sqrt(2)exp(5iπ/2)

One can generalize this to-

$$F(z,n) = (\sqrt{2})^n \exp(\frac{i\pi n}{4})$$

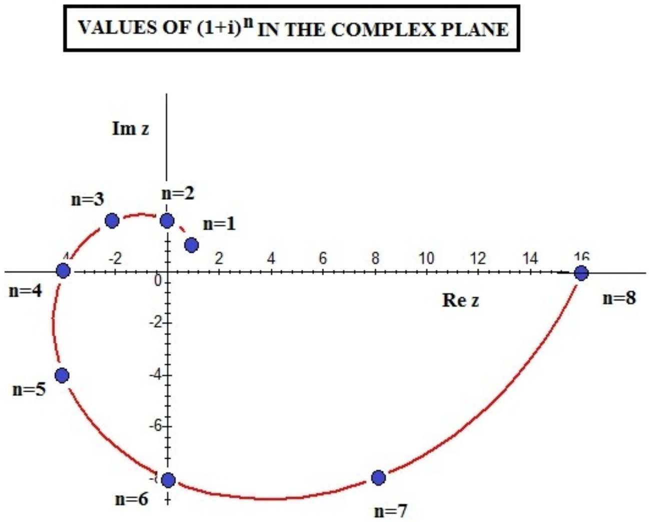
In the complex z plane these points correspond to-

$$r = (\sqrt{2})^n \quad and \quad \theta = \frac{\pi n}{4}$$

Eliminating n from these coordinates, we arrive at the well known Bernoulli (or logarithmic) Spiral-

$$\ln(r) = \left\{\frac{2\ln(2)}{\pi}\right\}\theta$$

This result shows that that the numbers F(z,n) all lie on this Bernoulli spiral. We demonstrate this point with the following graph-



The values for $(1+i)^n$ for integer n are shown as blue dots while the red curve is a continuous Bernoulli Spiral for the same function. This is an interesting result since it means that we can now locate any positive integer n as a unique point-

$$[r, \theta]=[\ \mathrm{sqrt}(2)^{\ n}, n\pi/4]$$

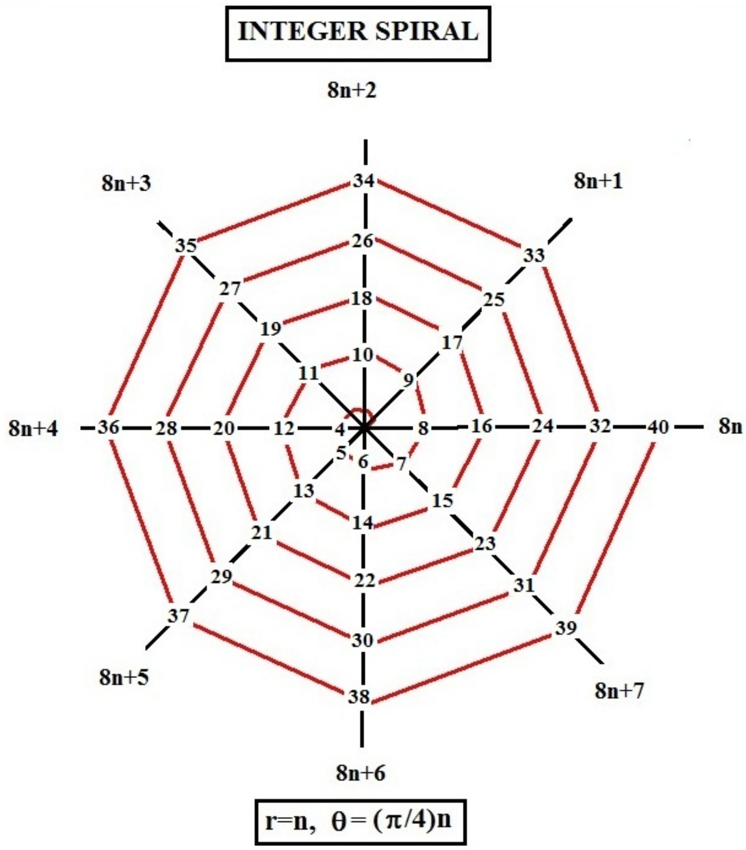
within the z plane. One notices, however, that the radial position of n, which equals |z|, grows very rapidly with increasing n. To avoid this problem one needs to decrease the growth rate of r. One way to do this is to simply set r=1. This unfortunately places all integers on the same unit radius circle in the z plane. To get around this difficulty we suggested years ago (see http://www2.mae.ufl.edu/~uhk/MORPHING-ULAM.pdf) that one lets r=n and therefore looks at the alternate complex function-

$$F(z,n,m) = n\exp(\frac{i\pi n}{m})$$

, with m a chosen positive integer. The real and imaginary parts of this expression are-

$$x = n\cos(\frac{n\pi}{m}) \quad and \quad y = n\sin(\frac{n\pi}{m})$$

This is recognized to be the parametric plot for an Archimedes Spiral once m is fixed. The angular spacing between neighboring integers will be $\Delta\theta=\pi/m$. We have found that two of the most useful angular spacings correspond to m=4 and m=3. Here is a plot for F(z,n,4)-



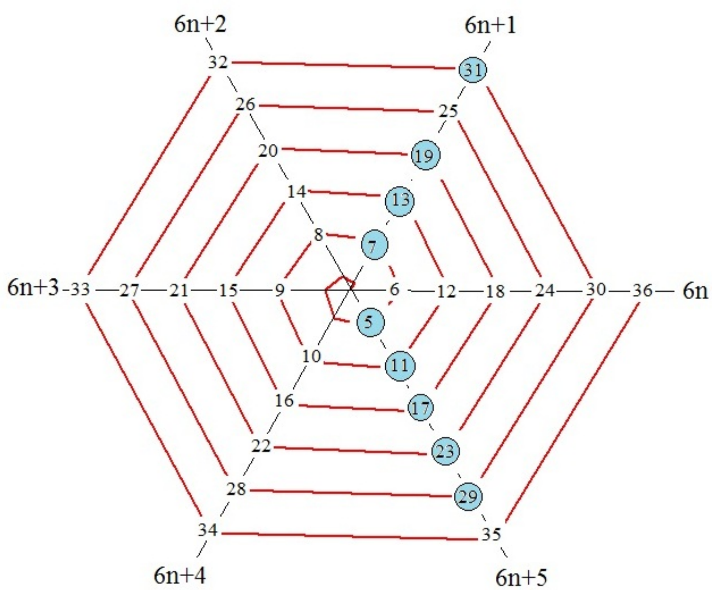
We call this figure an Integer Spiral for obvious reasons. It is superior to the classic Ulam spiral in that the even and odd integers are clearly separated along different radial lines. All even integers lie along either the x=Re(z) or y=Im(z) axis and separated from each other by a factor of eight for every turn of the spiral. All odd numbers fall along the two diagonals and are also separated by a value of eight between neighbors along the same line. It is easy to see along which line a number is located by just looking at n mod(8). Thus n=23789 has n mod(8)=5 and so lies along the diagonal line 8n+5 at the 2973rd turn of the spiral. Notice that we connect neighboring integers with just straight lines instead of a curved Archimedes spiral . This stems from the fact that we are dealing only with integer ns. With the exception of n=2, it is also clear that all prime numbers must lie along the diagonal lines, but clearly not all integers along these diagonals are primes. So n= 21, 27,33, and any number ending in 5 among an infinite number of others numbers are composites although they lie along the diagonals.

In our studies on prime numbers we have found that all prime numbers above p=3 must have the form 6n±1, although there are also an infinite number of composites of this form. This fact suggests that one might want to look at the complex function-

$$F[z,n,3]=n\ \exp(i\pi n/3)$$

to better separate composite from prime integers. We have done this and find the following integer spiral-

HEXAGONAL INTEGER SPIRAL AND THE LOCATION OF THE FIRST FEW Q PRIMES



The graph clearly shows that all primes(circled in blue) greater than p=3 lie along the two diagonal lines 6n+1 and 6n+5. There are no exceptions found. All even integers are found along the lines 6n, 6n+2, and 6n+4. The line 6n+3 has all odd integers separated by 6 from their neighbors. None of the odd numbers along this line can be prime! Let us demonstrate this last fact with the number-

n=83601109347231967215791541679014763977256487931

It has n mod(6)=3 and so is a composite number. My home PC is unable to factor this 47 digit long number into its prime components. This is the first integer spiral found by anyone which has all primes greater than two lie along just two lines in the z plane. It is far superior over attempts by others to use Ulam spirals where the prime numbers are found to be scattered all over the place.

There are certain groups of odd integers which can be represented by analytic expressions. These include the Mersenne Numbers and the Fermat Numbers defined, respectively, as-

$$M = 2^p - 1 \quad and \quad T = 2^{2^n} + 1 \quad with \quad p = prime$$

Some of these numbers are found to be primes. In the above hexagonal-spiral diagram the Mersenne Numbers all lie along the 6n+1 diagonal. The Fermat numbers all lie along the 6n+5 line. This is equivalent to saying –

M mod(6)=1 and T mod(6)=5

Finally, we have also given some thought to integer spirals using still larger values of m in the F(z,n,m) function. So, for instance, taking m=6, we get an integer spiral with twelve radial lines separated from each other by the angle $\Delta\theta=\pi/12$ rad=15 deg. Looking at the integers along these twelve radial lines, we find-

12n	12, 24, 36, 48, 60, 72, 84, 96,...
12n+1	**13**, 25, **37**, 49, **61**, **73**, 85, **97**,…
12n+2	14, 26, 38, 50, 62, 74, 86, 98, …
12n+3	15, 27, 39, 51, 63, 75, 87, 99,…
12n+4	16, 28, 40, 52, 64, 76, 88, 100,…
12n+5	**17**, **29**, **41**, **53**, 65, 77, **89**, **101,**…
12n+6	18, 30, 42, 54, 66, 78, 90, 102,…
12n+7	**19**, **31**, **43**, 55, **67**, **79**, 91, **103**,. ..
12n+8	20, 32, 44, 56, 68, 80, 92, 104,...
12n+9	21, 33, 45, 57, 69, 81, 93, 105,…
12n+10	22, 34, 46, 58, 70, 82, 94, 106
12n+11	**23**, 35, **47**, **59**, **71**, **83**, 95, **107**, …

This says prime numbers (shown in red) are found only along the four radial lines 12n+1, 12n+5, 12n+7, and 12n+11. So one can discard the remaining two radial lines with odd integers divisible by three when searching for primes. If we have the number n=123456789, than n mod(12)=9. This means n is composite. The number n=1873417 has n mod(12)=1. So the number may or may not be prime. An ifactor(n) operation with our computer shows that n is here composite with $1873417=(7^2)(13)(17)(173)$. All Mersenne Numbers Primes sdven or larger have the property that M mod (12)=7.

U,H,Kurzweg
Feb.2015