INTEGER SPIRAL

Several years ago we found a way to conveniently plot all positive integers as points along an Archimedean spiral. We want here to examine this problem in more detail. We begin by defining a complex function-

\[ f(N) = N \exp(i \frac{\pi}{4} N) = N \cos(\frac{\pi}{4} N) + iN \sin(\frac{\pi}{4} N) \]

One can generate a sequence of complex numbers \( f(N) \) by evaluating things over the range \( 0 < N < 120 \). These numbers are easy to generate by the MAPLE operation-

\[
\text{points}:= \{ \text{seq}([N*\cos(\pi/4*N), N*\sin(\pi/4*N)], N=0..120) \}:
\]

Next these 120 points are plotted in the complex plane x-y and we also superimpose the Archimedean Spiral \( r=(4/\pi)\theta \). The commands for doing this are-

\[ A:=\text{pointplot(points, symbol=circle, color=red, axes=none)}: \]

and

\[ B:=\text{plot}(4*N/\pi, N=0..30*\pi, coords=polar, color=blue): \]

followed by-

\[ \text{display}(A,B, scaling=constrained, title='INTEGER SPIRAL'); \]
The red circles represent the positive integers N while the blue spiral represents the Archimedes Spiral. Note that the integers N all are located at the intersection of the spiral and one of eight straight lines shown in black. All even numbers lie along the x and y axis while all odd numbers are located along the diagonals x=y or x=-y.

There are eight categories of numbers defined by the value of k in the number $N = 8n + k$, $k = 0..7$ and n an integer. The value of k is easily determined by dividing the integer N by 8 and then looking at the remainder. Thus we can rewrite the number-

$$N = 245791 = 8(30723) + 7$$

In modular arithmetic language this is the same as saying-

$$245791 \mod 8 = 7$$
It indicates that 245791 lies along the diagonal in the 4th quadrant at the 30723rd turn of the spiral. The mod operation is built into most canned mathematics programs such as MAPLE or MATHEMATICA.

Two numbers lying along the same straight line will always be separated from each other by multiples of 8. Also note that (except for N=2) all prime numbers lie along one of the four diagonals. This fact has implications for the meaning of the Ulam prime spiral which (as we have shown earlier) is really just a morphed version of the present integer spiral and its pattern really says nothing more than that all prime numbers are odd.

Two important set of numbers which produce integer values are the Fermat Numbers and the Mersenne Numbers. They are defined as -

\[ F(n) = 2^{2^n} + 1 \quad \text{and} \quad M(n) = 2^{2^n+1} - 1 \quad \text{for} \quad n = 1, 2, 3, 4... \]

These numbers will be prime for certain values of n. For example, F(3)=257 and M(7)=127 are prime. However, F(5)=4294967297 and M(11)=2047 are composite.

To find where the numbers F(n) and M(n) are located within the x-y plane one simply performs the mod operation. For F(n) this produces a remainder of one and means that all Fermat numbers lie along the diagonal in the first quadrant. For example,-

\[ F(4) = 2^{16} + 1 = 65537 = 8(9192) + 1 \]

and

\[ F(6) = 2^{64} + 1 = 8(2305843009213693952) + 1 \]

Note also that, as expected, the difference F(6)-F(4) is a multiple of 8.

All Mersenne Numbers above n=1 show a remainder of 7 after division by 8. Thus Mersenne numbers, including those which are prime, all lie along the diagonal in the 4th quadrant and differ from each other by multiples of eight. Thus-

\[ M(6) = 2^{13} - 1 = 8(1023) + 7 , M(7) = 8(4095) + 7 \]

and

\[ M(8) = 2^{17} - 1 = 8(16383) + 7 , M(9) = 2^{19} - 1 = 8(65535) + 7 \]
We also observe that all these Mersenne numbers end in either 1 or 7 when expressing things in a decimal number system. In binary these numbers are represented by a series of all 1s and no 0s, and thus all end in 1.

One may use modular arithmetic to carry out basic operations involving integers \( N \). Take the case of addition using the two specific numbers-

\[
N_3 = N_1 + N_2 = 3427 + 6893 = \{8(428) + 3\} + \{8(861) + 5\} = 8(1289) + 8 = 10320 \quad \text{so} \quad 10320 \mod 8 = 0
\]

It shows that the sum \( N_3 \) lies along the positive x axis at the 1290\(^{\text{th}}\) turn of the spiral.

If we perform a product operation of these numbers, we find-

\[
N_4 = N_1 \cdot N_2 = \{8(428) + 3\} \{8(861) + 5\} = 64(428)9861 + [40(428) + 24(861)] + (3)(5) \mod(8) = 23622311 = 8(2952788) + 7 \quad \text{and} \quad 23622311 \mod 8 = 7
\]

The product thus lies along the diagonal in the 4\(^{\text{th}}\) quadrant. We can also ask where will the large number \( N=45698210358229 \) lie. The answer is along the diagonal in the 3\(^{\text{rd}}\) quadrant since-

\[
45698210358229 \mod 8 = 5
\]

The 21 digit prime number \( N=3^{17}+5^{29}+9 \) yields-

\[
186264514923224843297 \mod 8 = 1
\]

and so lies along the diagonal in the 1\(^{\text{st}}\) quadrant.

With the mod 8 operation one can also quickly tell in which quadrant a number lies. Take, for example, the product-

\[
234567 \cdot 692314 \rightarrow 7 \cdot 2 = 14 \rightarrow 6
\]

It lies along the negative y axis at \( y = -162394018038 \).

Finally we look at what happens when \( N \) is non-integer. Keeping the same definition used for integer \( N \), we can express, for example, the square root of two as-
\[ \sqrt{2}\{\cos\left(\frac{\pi \sqrt{2}}{4}\right) + \sin\left(\frac{\pi \sqrt{2}}{4}\right)\} = 0.627933223.. + i 1.267162131... \]

So that root two lies at a radial distance \( \sqrt{2} \) from the origin at angle 
\( \theta = \arctan(2.01798867) = 63.65 \) deg in the first quadrant. The neighboring number 
\( N = \frac{3}{2} \) will lie at radial distance \( 1.5 \) and angle \( 3\pi/8 \) \( \text{rad} = 67.5 \) deg = \( (90+45)/2 \) deg. 
Using the mod operation we can also pinpoint the location of a non-integer \( N \) such as \( N = 347.95 \) as follows-

\[ N = 347.95 = 348 \mod 8 - 0.05 \quad \text{with} \quad 348 \mod 8 = 4 \]

Thus \( N \) lies in the second quadrant at radial distance 347.95 from the origin at angle 
\( \theta = 79\pi/80 \text{rad} = 177.75 \) deg.

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