EVALUATING CERTAIN INTEGRALS EXTENDING FROM ZERO TO INFINITY

A little over two hundred years ago the famous German mathematician Karl Gauss came up with a method for evaluating elliptic integrals of the first kind $K(m)$ by what is now known as the AGM method. The essence of the method is the identity:

\[
K(m) = \frac{1}{\sqrt{1-m}} \int_{x=0}^{\infty} \frac{dx}{\sqrt{[1+x^2][(\frac{1}{\sqrt{1-m}})^2 + x^2]}} = \frac{1}{\sqrt{1-m}} \int_{0}^{\infty} \frac{dx}{M^2 + x^2} = \frac{\pi}{2M\sqrt{1-m}}
\]

where $M$ is the geometric and arithmetic mean of 1 and $1/\sqrt{1-m}$ obtained by continuous iterations. Thus if $m=0.5$ we have $M=1.198140234$ formed from 1 and $\sqrt{2}$. It yields $K(0.5)=1.8540746\ldots$ The problem of integrating the product term in the radical of the first integral is thus reduced essentially to finding the value of $M$ by algebraic methods.

It is natural to ask if this approach will work for other types of integrals with semi-infinite range. The quick answer is no as we will demonstrate below.

We begin with the related integral:

\[
I(a,b) = \int_{x=0}^{\infty} \frac{dx}{(a^2 + x^2)(b^2 + x^2)} = \frac{\pi}{2(a+b)ab}
\]

which has the exact value given. Note that the arithmetic mean is here $A_0=(a+b)/2$ and the geometric mean is $G_0=\sqrt{ab}$. So that we may write:

\[
I(a,b) = \frac{\pi}{4A_0G_0^2}
\]

If one now applies the iteration procedure of Gauss $n$ times to generate $A_n$ and $G_n$ the two quantities will approach the same value $M$. Unfortunately the resultant equivalent integral:

\[
J(a,b) = \frac{\pi}{4M^3}
\]

is close but does not match exactly the value of $I(a,b)$. So for this integral the AGM method fails. We can however work backwards and ask what value $N$ is required to make:

\[
\int_{x=0}^{\infty} \frac{dx}{(N^2 + x^2)^2} = \frac{\pi}{2(a+b)ab} = \frac{\pi}{4N^3}
\]

The answer is:
\[ N = \left(\frac{a+b}{2}\right)^{1/3} = \{A_0 G_0\}^{1/3} \]

If we next let \( a = 1 \) and \( b = 2 \) we find-

\[ I(1,2) = \int_0^\infty \frac{dx}{(3^{2/3} + x^2)^2} = \frac{\pi}{12} \]

In this case \( A_n = Q_n = 1.453679 \ldots \) as \( n \) gets large while \( 1^{3/3} = 1.442295 \ldots \) So the AGM method fails.

Also if we take \( a = 3 \) and \( b = 5 \) we get \( N = 60^{1/3} \) and have the integral-

\[ I(3,5) = \int_0^\infty \frac{dx}{(60^{2/3} + x^2)^2} = \frac{\pi}{240} \]

Here are some other related integrals whose value we know and which have a related form not recoverable by AGM methods-

\[ R(a,b) = \int_0^\infty \frac{x dx}{(a^2 + x^2)(b^2 + x^2)} = \int_0^\infty \frac{dx}{\left[\frac{3}{\ln(4)} + x^2\right]^2} = \frac{\ln(2)}{3} \]

\[ S(a,b) = \int_0^\infty \frac{x^2 dx}{(a^2 + x^2)(b^2 + x^2)} = \int_0^\infty \frac{x^2 dx}{\left[(a+b)^2 / 2 + x^2\right]} = \frac{\pi}{2(a+b)} \]

\[ T(a,b) = \int_0^\infty \frac{(c^2 + x^2) dx}{(a^2 + x^2)(b^2 + x^2)} = c^2 I(a,b) + S(a,b) = \frac{\pi}{4(a+b)ab} (c^2 + ab) \]

We have shown via the above examples that AGM methods work only when the product term \((a^2 + x^2)(b^2 + x^2)\) appears as a square-root in the denominator of the type of integral considered by Gauss. Removing the radical form leads to other convergent solutions as long as the maximum power of \( x \) in the denominator of the integral exceeds the power of \( x \) in the numerator. Use of \( A_\infty = G_\infty \) seems to work only for integrals of the type capable of generating elliptic functions of the first kind \( K(m) \) which means a radical in the denominator of the integral.
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