ITERATION PROCEDURES FOR FINDING ROOTS, EXP(X), ARCTAN(X), AND LN(X)

We begin by looking at roots of numbers and how these are quickly obtained by iteration methods. Our starting point is the Taylor Series Expansion for \( x^{1/p} \) about \( x=a \). One has -

\[
x^{1/p} = a^{1/p} + \frac{a^{1/p}(x-a)}{l! \cdot p a} + \frac{a^{1/p}((1-1)(x-a)^2)}{2! \cdot p a^2} + \ldots
\]

If we now terminate the series after the first two terms and set-

\[ a^{1/p}=x[n] \text{ and } x^{1/p}=x[n+1] \]

we obtain the iteration algorithm-

\[ x[n+1] = x[n] + \frac{x[n] (N - x[n]^p)}{px[n]^p} \]

where \( N \) is the number whose \( p \)th root we are interested in finding. Let us start with the sqrt(2) and the initial guess \( x[0]=b=1 \), we have-

\[ x[1] = 1 + \frac{(2-1)}{2} = \frac{3}{2} \quad \text{and} \quad x[2] = \frac{3}{2} + \frac{(2-2.25)}{2(1.5)} = \frac{17}{12} \]

Instead of continuing further hand calculations, we next invoke the computer command-

\[ x[0] := 1; \quad \text{for n from 0 to 6 do } x[n+1] := evalf(x[n]+(2-x[n]^2)/x[n]^2, 50) \text{ od;} \]

This quickly leads to approximations \( x[1] \) through \( x[7] \) expressed as fifty digit long numbers. Already for \( x[6] \) we have a 48 place accurate result given by-

\[ x[6] := 1.41421356237309504880168872420969807856967187537 \]

Consider next finding a highly accurate approximation for the Golden Ratio \( \phi=(1+\sqrt{5})/2 \). Here one needs to first find the value \( \sqrt{5} \) and then with a little bookkeeping obtain the value for \( \phi \). The \( \sqrt{5} \) is generated by the command-

\[ x[0] := 2; \quad \text{for n from 0 to 10 do } x[n+1] := evalf(x[n]+(5-x[n]^2)/(2*x[n]), 80) \text{ od;} \]

and at \( x[8] \) has the eighty digit accurate result-
Thus the Golden Ratio, good to eighty places of accuracy, reads:

\[ \phi = 1.6180339887498948482045868343656381177203091798057628621354486227052604628189025 \]

We can also quickly calculate other roots of N. For example the third root of N=7 follows from the iteration:

\[ x[0] = 2 \quad \text{with} \quad x[n+1] = x[n] + \frac{(7 - x[n]^3)}{3x[n]^2} \]

We find \( x[1] = \frac{23}{12} \) and our computer command reads:

\[
x[0]:=2;\quad \text{for } n \text{ from 0 to 10 do } x[n+1]:=\text{evalf}(x[n]+(7-x[n]^3)/(3*x[n]^2),50)\od;
\]

Already at the 8th iteration we have the fifty place accurate result:

\[ 7^{\frac{1}{3}} \approx x[8] := 1.9129311827723891011991168395487602828624390503459 \]

Besides the powers of N, most other functions such as \( \exp(x) \), \( \sin(x) \), \( \cosh(x) \), etc can also be expanded about a neighboring point \( x=a \) to obtain an iteration formula for quick approximations of these functions at specified values of x. Take first the exponential function expanded about \( x=a \). The first two terms of a Taylor Series allows one to write:

\[ \exp(x) = \exp(a)(1 + x - a) \]

So the iterative formula becomes:

\[ x[0]=1 \quad \text{with} \quad x[n+1]=x[n](1+x-\ln(x[n])) \]

Choosing \( x=1 \) we find that just six iterations produce the 26 digit accurate result:

\[ \exp(1) \approx x[6] = 2.7182818284590452353602874 \]

You will note that we required a knowledge of the value of \( \ln(x[n]) \) to get this result. To avoid this difficulty one could instead go to the full McLaurin series for \( \exp(x) \), namely,
\[
\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

and then write down the partial sums \( S[0] = 1, S[1] = 1 + x \) and carry out the iteration –

\[
S[n + 1] = S[n] + \frac{x^n}{n!}
\]

for a given value of \( x \). At \( x = 1 \) the tenth iteration yields the 19 digit accurate result-

\[
S[20] = 2.7182818284590452355
\]

Notice this approach is essentially equivalent to adding one extra term to the infinite series at each iteration and so is typically slower to converge than the other approach used above, provided this is possible. Some improvement is offered by telescoping the infinite series to-

\[
\exp(x) = \sum_{n=0}^{\infty} \frac{(2n + 1 + x)x^{2n}}{(2n + 1)!}
\]

The iterative formula at \( x = 1 \) then becomes-

\[
S[0] = 2 \text{ and } S[n + 1] = S[n] + \frac{(2n + 4)}{(2n + 3)!}
\]

Here \( S[10] \) produces a 20 digit accurate approximation for \( \exp(1) \).

The next function we want to look at is the sine integral function at \( x = 1 \). Here we have the definition-

\[
\sinh(x) = \int_{t=0}^{1} \frac{\sin(t)}{t} dt = \sum_{k=0}^{\infty} \frac{(-1)^n}{(2n + 1)(2n + 1)!}
\]

This yields the partial sums \( S[0] = 1, S[1] = 17/18, \) and \( S[2] = 1703/1800 \) or more generally the partial sum iteration-

\[
S[n + 1] = S[n] + \frac{(-1)^{n+1}}{(2n + 3)(2n + 3)!} \quad \text{with} \quad S[0] = 1
\]

Running through the first ten iterations we obtain the 23 digit accurate approximation-

\[
\sinh(1) \approx x[10] = 0.9460830703671830149413550
\]
The convergence rate is this time better than it was for the exponential function when using the partial sum approach for the non-telescoped case. Things will converge even faster when one looks at $\text{Si}(x)$ for $x<1$.

The partial sum iterative approach for evaluating certain functions can involve some difficulties. For example, $\arctan(x)$ has the form:

$$\arctan(x) = \int \frac{dx}{1 + x^2} = \sum_{k=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}$$

Here the denominator in the infinite series increases only slowly in size with increasing $n$ and hence the series converges very slowly unless $x<<1$. Here our iterative formula reads:

$$S[0] = x \quad \text{with} \quad S[n+1] = S[n] + \frac{(-1)^n x^{2n+1}}{(2n+1)}$$

To get this iteration to quickly converge will require that $x<<1$. This fact is one of the reasons one looks for $\arctan$ formulas where $x=1/N$ has large values of $N$ when using multiple term $\arctan$ expansions for calculating $\pi$. A good example of such a rapidly converging $\arctan$ formula is our own four term version:

$$\pi/4 = 12\arctan(1/38) + 20\arctan(1/57) + 7\arctan(1/239) + 24\arctan(1/268)$$

We can evaluate the various $\arctan$ terms in this last expression using the iteration:

$$x[0] = -1 \quad \text{with} \quad x[n+1] = x[n] + \cos(x[n])^2 \left( \frac{1}{N} - \tan(x[n]) \right)$$

which may also be written as:

$$x[n+1] = x[n] + \frac{1}{N} (\cos(x[n]))^2 - \frac{1}{2} \sin(2x[n]) \quad \text{with} \quad x[0] = \frac{1}{N}$$

For $N=38$, we get the 99 digit accurate result:

$$\arctan(1/38) \approx x[9] := 0.0263097172529221921299087309613372197997511424762478704438187406145123750095424878484680313525160902$$

after just nine iterations. Note that this evaluation required knowledge of $\cos(x[n])$ and $\sin(x[n])$ which both are represented by rapidly converging series. The approach does not
require the taking of roots of quantities as needed in AGM methods for finding \( \pi \). We point out that one can iterate directly the two term Machin Equation-

\[
\pi = 16 \arctan(1/5) - 4 \arctan(1/239)
\]

using the above iteration formula for \( \arctan(1/N) \) to find a highly accurate value for \( \pi \) with even less effort. Taking things out to just \( x[5] \) produces the 90 digit approximation-

\[
\pi \approx x[5] = 3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803483
\]

We complete out iteration discussions by examining the value of \( \ln(2) \). This constant may be written as-

\[
\ln(2) = \int_{t=0}^{1} \frac{dt}{1+t} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots
\]

This represents an extremely slowly convergent infinite series. How does one improve the convergence? One way is to go back to the Taylor series approach for \( \ln(1+x) \). If \( x \approx a \), we have-

\[
\ln(1+x) = \ln(1+a) + (x-a)/(1+a)
\]

Choosing \( z[n+1] = \ln(1+x) \) and \( z[n] = \ln(1+a) \), we get for \( x=1 \) the algorithm-

\[
z[n+1] = z[n] + 2 \exp(-z[n]) - 1 \quad \text{with} \quad z[0] = 1
\]

We have run this iteration formula through \( z[7] \) starting with \( z[0]=1 \) and have recorded the results in the following figure-
One observes that the accuracy approximately doubles in the number of correct integers per iteration. For \( z[7] \) we find \( \ln(2) \) given to 90 digit accuracy.

We also tried an iteration approach for \( \ln(2) \) based on partial sums and the telescoped form-

\[
\ln \left( \frac{1 + x}{1 - x} \right) = \ln(1 + x) - \ln(1 - x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{(k + 1)} \left[ 1 + (-1)^k \right]
\]

For \( x = 1/3 \), it produces the partial sum iteration -

\[
S[n + 1] = S[n] + \frac{2}{(2n + 3)3^{2n+3}} \quad \text{with} \quad S[0] = 2/3
\]

Carrying out ten iterations produces an 11 place accurate result. Thus, this last approach is not a very effective iteration approach for quickly finding highly accurate values for \( \ln(2) \). Its only advantage is that no knowledge of \( \exp(-z[n]) \) is required.

U.H.Kurzweg
Gainesville, Florida
December 2012