ITERATION OF FUNCTIONS

A typical continuous function $f(x)$ may always be expanded in a Taylor series:

$$f(x) = f(a) + f'(a)(x-a)/1! + f''(a)(x-a)^2/2! + ...$$

When $x$ and ‘a’ lie close enough to each other then the linear approximation:

$$f(x) \approx f(a) + f'(a)(x-a) \text{ subject to } z[1]=\text{const.}$$

holds provided that $f'(a)$ does not vanish. Now if we know $f(a)$ and its derivative, then $f(x)$ can be estimated. A good way to express this last result is by an iteration approach where we set $z[n+1]=f(x)$ and $z[n]=f(a)$. Writing out the resultant iteration form we have:

$$z[n+1] = z[n] + f'(a_n)(x-a_n) \text{ with } f(a_n) = z[n] \text{ and } z[1] = f(a) = \text{const.}$$

The value of $f(x)$ is reached when $n=\infty$.

Let us demonstrate things for the $f(x)=\sqrt{x}$ where $f'(x)=1/2\sqrt{x}$ and let us consider $x=2$. This produces the iteration formula

$$z[n+1] = z[n] + \frac{(2-a)}{2\sqrt{a}} = \frac{2 + z[n]^2}{2z[n]} \text{ subject to } z[1] = 1$$

Substituting in, we find $z[2]=3/2$, $z[3]=17/12$, and $z[4]=577/408=1.1414215$. So the fourth iteration is already accurate to six decimal places for $\sqrt{2}$. One can easily program this iteration formula as follows:

$$z[1]=1; \text{ for } n \text{ from } 1 \text{ to } 8 \text{ do evalf}((2+z[n]^2)/(2*z[n])) \text{ od;}$$

This produces-
Here the iterations are taken out to the point where one first notices a departure from the exact value of sqrt(2). At the eighth iteration we already have a 90 decimal point accuracy. Note that in this iteration procedure the first few iterates are not very accurate but then the accuracy takes off rapidly as n increases further.

We can also run an iteration for the Golden Ratio using essentially the same approach. This time we have:

\[ z[n + 1] = \frac{(z[n]^2 + 1)}{2z[n] - 1} \quad subject \quad to \quad z[1] = 2 \]

It produces the result:

**Determining the Golden Ratio by Iteration**

\[ z_1 := 2 \]
\[ z_2 := 1.6 \]
\[ z_3 := 1.61 \]
\[ z_4 := 1.61803 \]
\[ z_5 := 1.618033988749 \]
\[ z_6 := 1.61803398874989484820458683 \]
\[ z_7 := 1.61803398874989484820458683436563811772030917980576286 \]
\[ z_8 := 1.6180339887498948482045868343656381177203091798057628621354486227052604628189024970720720 \]

using the computer command:

\[ z[1]:=2; \text{ for } n \text{ from 1 to 8 do } z[n+1]:=(1+z[n]^2)/(2*z[n]-1),90) \text{ od;} \]
So \( z[8] \) produces a 90 digit accurate result. Usually the Golden Ratio is designated by \( \phi \). Its inverse \( 1/\phi = \sqrt{5} - 1 \) is \( \phi - 1 = 0.61803398 \). Also we have \( \phi^2 = \phi - 1 \). This algebraic equation can be used to find all higher powers of the Golden Ratio.

As a third example consider the logarithmic function \( f(x) = \ln \left( \frac{1+x}{1-x} \right) \) which has \( df(x)/dx = 2/(1-x^2) \). Here we set \( z[n+1] = \ln \left( \frac{1+x}{1-x} \right) \) and \( z[n] = \ln \left( \frac{1+a}{1-a} \right) \). We set \( x = 1/3 \) so that we will get \( f(1/3) = \ln(2) \). The iteration formula becomes:

\[
z[n + 1] = z[n] + \frac{2}{1 - a^2} (1/3 - a) \quad \text{subject to} \quad z[1] = 0.5
\]

and the definition –

\[
a = \frac{\exp(z[n]) - 1}{\exp(z[n]) + 1}
\]

So the print-out for \( \ln(2) \) becomes –

**FIRST EIGHT ITERATES FOR LN(2)**

\[
\begin{align*}
z_1 &= \frac{1}{2} \\
z_2 &= 0.6881 \\
z_3 &= 0.69314 \\
z_4 &= 0.69314718055 \\
z_5 &= 0.69314718055994530941723 \\
z_6 &= 0.6931471805599453094172321214581765680755001343 \\
z_7 &= 0.6931471805599453094172321214581765680755001343602552541206800094933936219696947156058633
\end{align*}
\]

Again the approximation accuracy increases slowly for small \( n \) but then accelerates as \( n \) gets large. The iteration \( z[7] \) yields \( \ln(2) \) to 88 place accuracy.

So far we have encountered no difficulties using a two term Taylor series to construct our iteration formulas for a given \( f(x) \). Unfortunately this approach works well only as long as the derivative of \( f(a) \) leads to a function which contains no terms which are more difficult to evaluate than the function \( f(x) \) itself. One such case of many arises in the evaluation of \( \exp(1) \). The above general iteration form yields:

\[
\exp(x) = \exp(a) + \exp(a)(x-a)
\]

This reads –
\[ z[n+1] = z[n] \cdot \{1+x-a\} \quad \text{with} \quad a = \ln(z[n]) \]

So if we are interested in \( \exp(1) \) you can set \( x=1 \) in the iteration formula and start with \( z[1]=1 \). Iterating this formula we get \( z[1]=1, z[2]=2, z[3]=4-2\ln(2) \). We see that the evaluation of \( \exp(1) \) will require knowledge of the natural logarithms which are more difficult to evaluate accurately than the original \( f(x)=\exp(x) \). This requires one seek another route to evaluate \( \exp(x) \) accurately by iteration.

One way to do this is to consider the infinite series for \( \exp(1) \) taking two terms at a time. It allows us to write the following iteration form for \( \exp(1) = 2.718281828459045… \)

\[
T[n+1] = T[n] + \frac{(2n+1)!(2n)!}{(2n+1)!(2n)!} \quad \text{subject to} \quad T[1] = 2
\]

The convergence rate for this expression is a bit slower than what we found in earlier examples involving powers of numbers but still better than trying to go the logarithm route. Here is the output for iterates \( T[25] \) through \( T[34] \):

<table>
<thead>
<tr>
<th>Iteration Results for ( \exp(1) ) Showing Iterates ( T[25] ) Through ( T[34] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_{25} ) = 2.71828182845904525360287471352662497757247093699995957496696967627</td>
</tr>
<tr>
<td>( T_{26} ) = 2.718281828459045253602874713526624977572470936999959574966969676277240</td>
</tr>
<tr>
<td>( T_{27} ) = 2.718281828459045253602874713526624977572470936999959574966969676277240766</td>
</tr>
<tr>
<td>( T_{28} ) = 2.7182818284590452536028747135266249775724709369999595749669696762772407663035</td>
</tr>
<tr>
<td>( T_{29} ) = 2.71828182845904525360287471352662497757247093699995957496696967627724076630353547</td>
</tr>
<tr>
<td>( T_{30} ) = 2.71828182845904525360287471352662497757247093699995957496696967627724076630353547594</td>
</tr>
<tr>
<td>( T_{31} ) = 2.718281828459045253602874713526624977572470936999959574966969676277240766303535475945713</td>
</tr>
<tr>
<td>( T_{32} ) = 2.718281828459045253602874713526624977572470936999959574966969676277240766303535475945713821</td>
</tr>
<tr>
<td>( T_{33} ) = 2.718281828459045253602874713526624977572470936999959574966969676277240766303535475945713821460</td>
</tr>
<tr>
<td>( T_{34} ) = 2.718281828459045253602874713526624977572470936999959574966969676277240766303535475945713821460469</td>
</tr>
</tbody>
</table>

The error in stopping after \( n \) iterations is approximately \( 1/ (2n)! \). That is, the error in \( T[34] \) will be about \( 1/68! = 0.40322 \times 10^{-96} \). Several years back I invented a new mnemonic which allows one to remember the first 33 digits of \( \exp(1) \). It reads-

2.7+Adrew Jackson twice(18281828)+45 deg triangle(45-90-45)-Fibonacci three(235)-full circle(360)-year before crash(28)+Boeing’s best(747)+end of black death in Europe(1352)+route going west(66).

Spelled out this says-
exp(1)=2.71828182845904523536028747135266

Another important mathematical constant is the Catalan constant defined as –

\[
CAT = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots = 0.915965594401
\]

Its series is very slow to converge having no factorials in the denominator of its series. We can seek an iteration formula for this constant obtained by the following manipulations. First we recall that \(\pi^2/8\) represents the sum of a series differing from the CAT expansion only in that all its sign are positive. This means we can write-

\[
CAT = \frac{\pi^2}{8} - 2 \sum_{n=0}^{\infty} \frac{1}{(3+4n)^2} = \frac{\pi^2}{8} - \frac{2}{9} - 2 \sum_{n=1}^{\infty} \frac{1}{(3+4n)^2}
\]

Although the series in this last expression is also very slow to converge, we can make use of the Laplace transform identity-

\[
\sum_{n=1}^{\infty} F[n] = \int_{0}^{\infty} f(t) \exp(-nt) \frac{dt}{\exp(t) - 1}
\]

where Laplace(f(t))=F[n]. So on setting F[n]=1/(3+4n)^2, we get the identity-

\[
CAT = \frac{\pi^2}{8} - \frac{2}{9} - \frac{1}{8} \int_{0}^{\infty} t \exp(-3t/4) \frac{dt}{\exp(t) - 1} = \frac{(\pi^2 - \Psi(1,3/4))}{8}
\]

Here \(\Psi(1,x)\) is the digamma function defined as-

\[
\Psi(1,x) = \frac{d^2}{dx^2} \{\ln(\Gamma(x))\} \quad \text{where} \quad \Gamma(x) \quad \text{is the Gamma Function}
\]

From the above we have the important identity-

\[
CAT = \frac{\pi^2 - \Psi(1,3/4)}{8}
\]

The value of CAT may now be determined by iterating \(\Psi(1,3/4)\). We let \(z[n+1]=\Psi(1,x)\) and \(z[n]=\Psi(1,a)\). This produces-
\[ z[n+1] = z[n] + (3/4 - a) \left( \frac{d(z[n])}{da} \right) \quad \text{with} \quad z[n] = \Psi(1,a) \quad \text{and} \]

\[ z[1] = \Psi(1,1) = \pi^2/6. \] If we let \( n \) approach infinity we have \( z[n+1] = z[n] \) and hence \( a = 3/4 \). Thus \( z[\infty] = \Psi(1,3/4) = 2.5418796476716064983976628804170782491205044129874… \)

This leaves us with –

\[ \text{CAT} = 0.91596559417721901505460351493238411077414937428167… \]

We are not sure what procedure our MAPLE program uses to calculate \( \Psi(1,3/4) \). It is assumed that it does this more easily than calculating CAT directly.

As a final iteration consider finding the value of \( \pi/4 \) using the definition \( \pi/4 = \arctan(1) \). One has here the approximation-

\[ \arctan(x) = \arctan(a) + \frac{1}{1+a^2} (1-a) \]

On letting \( z[n+1] = \arctan(x) \) and \( z[n] = \arctan(a) \) we get-

\[ z[n+1] \approx z[n] + \frac{1}{1+\tan(z[n])^2} (1-\tan(z[n])) \quad \text{subject to} \quad z[1] = 1 \]

The first eight terms of this iteration formula tread-

### FIRST EIGHT ITERATIONS FOR PI/4

\[ \begin{align*}
    z_1 &= 1 \\
    z_2 &= .8 \\
    z_3 &= .78 \\
    z_4 &= .785 \\
    z_5 &= .78598163 \\
    z_6 &= .78598162339744830961 \\
    z_7 &= .785981623397448309615660845819875721049 \\
    z_8 &= .785981623397448309615660845819875721049292349843776455243736148076954101571552249
\end{align*} \]

**Digits taken to where iteration first departs from exact value**

81 digit accuracy at \( z[8] \)
Note how rapidly things converge toward the exact answer. Here very accurate values for \( \tan(x) \) are known since its components \( \sin(x) \) and \( \cos(x) \) both contain factorials in their denominators and hence are known quite accurately.

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