PROPERTIES OF THE LAMBERT FUNCTION W(z)

The first order, non-linear, ODE-

$$\frac{dW(z)}{dz} = \frac{\exp[-W(z)]}{[1 + W(z)]} \quad \text{subject to } W(0) = 0$$

can be solved by the simple integration-

$$\int_0^z dW(z) = \int_0^z [1 + W(z)] \exp[1 + W(z)] dW(z)$$

to yield the implicit solution-

$$z = W(z) \exp[W(z)]$$

where W(z) is the Lambert function.

One can expand this function in a Taylor series-

$$W(z) = W(0) + \frac{dW(0)}{dz} z + \frac{d^2W(0)}{dz^2} \frac{z^2}{2!} + \ldots$$

to obtain-

$$W(z) = z - z^2 + \frac{3z^3}{2} - \frac{8z^4}{3} + O(z^5) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n$$

A plot of W(z) for z=x in the range -0.3678<x<4 follows-
There are several direct applications of the Lambert function. One of the better known of these is in finding the limit of an iteration connected with the tetration $N^N(N^N(N^N(N \ldots$). Tetration of a number $N$ can be represented by the iteration:

$$a[n+1] = N^{a[n]} = \exp(a[n]\ln N) \text{ with } a[0] = N$$

For this iteration to converge one must have that:

$$a[\infty]\exp(a[\infty]\ln(\frac{1}{N})) = 1$$

which is equivalent to:

$$a[\infty] = \frac{W(\ln(\frac{1}{N}))}{\ln(\frac{1}{N})}$$

Thus one has that tetration for $N=a[0]=i$ yields $a[\infty]=0.43828293..+i0.36059247..$
Another place where the Lambert function is encountered is in the solution of the difference equation-

\[ \frac{dx(t)}{dt} = c \ x(t - 1) \]

We try \( x(t) = \exp(b \ t) \) to yield-

\[ b = c \ \exp(-b) \ or \ equivalent \ b = W(c) \]

so that the equation yields the solution-

\[ x(t) = \exp[ W(c) \ t ] \]

Also one can find certain values of \( z \) for which \( W(z) \) assumes simple closed forms. Start with the function-

\[ F(z) = \frac{W(\ln(z))}{\ln(z)} = \exp(-W(\ln(z))) \]

We find this function has the exact values \( F[1/\sqrt{2}] = 2, F[1] = 1, \) and \( F[4] = 0.5. \) From these results one can infer, for example, that-

\[ W(2 \ln 2) = \ln(2) \ and \ W[\ln(\frac{1}{\sqrt{2}})] = -\ln(2) \]

This last result in turn suggests one try \( W[\ln(a)] = \ln(b). \) This leads to-

\[ W[\ln(a)] \exp W[\ln(a)] = \ln(b) \exp(\ln(b)) = \ln(a) \]

from which follows that \( a = b^b \) so that we have the identities-

\[ W(b \ln b) = \ln(b) \ and \ W[c \exp(c)] = c \]
where \( e = \ln(b) \). From these last identities follow the equalities:

\[
W[\exp(1)] = 1, \ W[\exp(-1)] = -1, \ W\left(\frac{-\pi}{2}\right) = i\frac{\pi}{2}
\]

Also by setting \( z = i \), we obtain the identity-

\[
\pi = -2i\{W(i) + \ln[W(i)]\}
\]