OBTAINING THE LIMITS FOR A FUNCTION AT ANY POINT

As those of you, exposed to elementary calculus in your high school or your freshman college years, learned that any differentiable functions given by the quotient $F(x)=f(x)/g(x)$ can be written in a Taylor series expansion about $x=x_0$ given by-

$$F(x) = \left\{ \frac{f(x_0) + f'(x_0)(x-x_0) + f''(x_0)(x-x_0)^2 / 2! + \cdots}{g(x_0) + g'(x_0)(x-x_0) + g''(x_0)(x-x_0)^2 / 2! + \cdots} \right\}$$

From this expansion one can determine at once the limiting value of $F(x)$ at $x_0$. Often this ratio will have zero value for the zeroth derivative and higher of $f(x_0)$ and $g(x_0)$. Among the limits many of you will be familiar with are-

$$\lim_{x \to 0} \left\{ \frac{\sin(x)}{x} \right\} = 1 \quad and \quad \lim_{x \to \infty} \left\{ \left(1 + \frac{1}{x}\right)^x \right\} = \exp(1)$$

To get the first limit at $x=0$, one writes-

$$F(0) = \lim_{x \to 0} \left\{ \frac{\sin(0) + \cos(0)x + \cdots}{x} \right\} = 1$$

A plot of $\sin(x)/x$ versus $x$ follows-
It nicely confirms the limit of \( \frac{\sin(x)}{x} = 1 \) at \( x=0 \). If we look at some other limiting point, say \( x_0=\frac{\pi}{2} \), we get-

\[
\lim_{x \to (\pi/2)} \left\{ \frac{1}{\pi/2} \right\} = \frac{2}{\pi}
\]

The first zero of \( \frac{\sin(x)}{x} \) occurs for \( \tan(x)=0 \) so \( x=\pm \pi \).

The other limit at \( x=\infty \) is gotten by first setting \( z=1/x \) to get-

\[
F(0) = \lim_{z \to 0} \left\{ (1+z)^{1/z} \right\} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = \exp(1)
\]

Note that in this second case we had no quotient but were able to expand \( (1+z)^{1/z} \) in a Taylor series about \( z=0 \).

Consider next the function -

\[ F(x) = x \ln(x) \]

which as first glance appears indeterminate at \( x=0 \). It however has the following limiting value-

\[
\lim_{x \to 0} \left\{ \frac{\ln(x)}{1/x} \right\} = \lim_{z \to \infty} \left\{ \frac{-\ln(z)}{z} \right\} = 0
\]

Also \( x \ln(x) \) has a second interesting limit at \( x=1 \). There we find that –

\[
\lim_{x \to 1} \left\{ \frac{\ln(x)}{1/x} \right\} = 0
\]

A plot of \( x \ln(x) \) showing these two limits at \( x=0 \) and \( x=1 \) follows-
Very often one finds that both the zeroth and some of the lower derivatives for both \( f(x) \) and \( g(x) \) vanish at \( x = x_0 \). In that case one has to move to higher derivative terms to find a limit. Take the case of-

\[
f(x) = x^5 \sinh(x) = x^6 + \left(\frac{1}{6}\right)x^8 + O(x^{10})
\]

and

\[
g(x) = \cosh(x^3) - 1 = \left(\frac{1}{2}\right)x^6 + O(x^9)
\]

Here we have to go to the sixth derivative to show that the limit of the quotient \( f(x)/g(x) = 2 \). A graph of \( F(x) = f(x)/g(x) \) follows-
As final example of a limit consider the function-

\[ F(a, x) = \left(1 + \frac{1}{ax}\right)^x = 1 + \frac{x}{1!ax} + \frac{x(x-1)}{2!(ax)^2} + \frac{x(x-1)(x-2)}{3!(ax)^3} + \ldots \]

As \( x \to \infty \) this function has the form-

\[ F(a, \infty) = \sum_{k=0}^{\infty} \frac{1}{(ak)!} = 1 + \frac{1}{1!a} + \frac{1}{2!a^2} + \frac{1}{3!a^3} + \ldots = \exp\left(\frac{1}{a}\right) \]

which becomes \( e \) when \( a \) is set to one. The following is a plot of \( F(1, x) \) -
Note that the convergence toward $\exp(1)$ is quite slow reaching only about ninety percent of the $x$ equal infinite limit at $x=15$. With larger ‘a’ the series converges much faster. So, for instance, an expansion in the sum up through $k=15$ and taking $a=2$ produces $1.648721270701281468486507$ which is accurate to 26 places in $\exp(0.5)$. Thus we get the much more accurate result:

$$\exp(1) \approx \left( 1.648721270701281468486507 \right)^2$$

U.H.Kurzweg
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Gainesville, Florida