AREAS CREATED BY STRAIGHT LINES AND CIRCLES

The simplest way to create a closed area is to draw a closed loop such as a circle. This is followed by using a circle and a single line. Next come two lines and a circle or two circles and a straight line. The generated areas become progressively more complicated as the number of circles and lines, which are made to intersect each other, become larger. It is our purpose here to find some of these areas without the use of calculus.

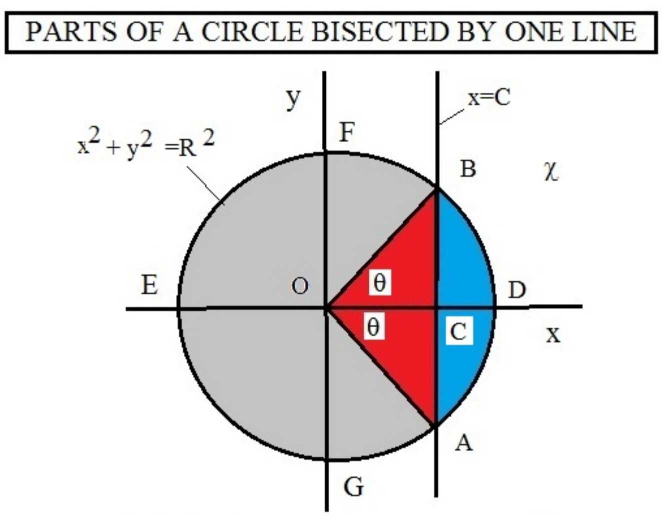
Our starting point is a simple circle given by the equation-

$$(x-a)^2+(y-b)^2=R^2$$

This circle is centered at [x,y]=[a,b] and has radius R. It encloses the area πR^2 and has a circumference of $2\pi R$. Any straight line in 2D reads-

$$y=n+mx$$

It has a slope of m and intersects the x axis at –n/m. Let us begin by determining the areas created by a radius R circle centered at the origin and a vertical line x=C, where |C|<R. Here is the picture-



OA=OB=radius R , A-D-B-C = segment , A-D-B-O = sector
A-C-B=cord , A-D-B =arc

By simple geometry and the Pythagorean Theorem we have the following-

$$ArcLength\ ADB = 2R\theta = 2R\arcsin\left\{\frac{\sqrt{R^2 - C^2}}{R}\right\}$$

$$Length\ of\ \ Cord\ \ AB = 2\sqrt{R^2 - C^2}$$

$$Sector\ \ AODB = (\pi R^2)(\frac{2\theta}{2\pi}) = R^2\theta = R^2\arcsin(\frac{\sqrt{R^2 - C^2}}{R})$$

$$Area\ Triangle\ ABO = C\sqrt{R^2 - C^2}$$

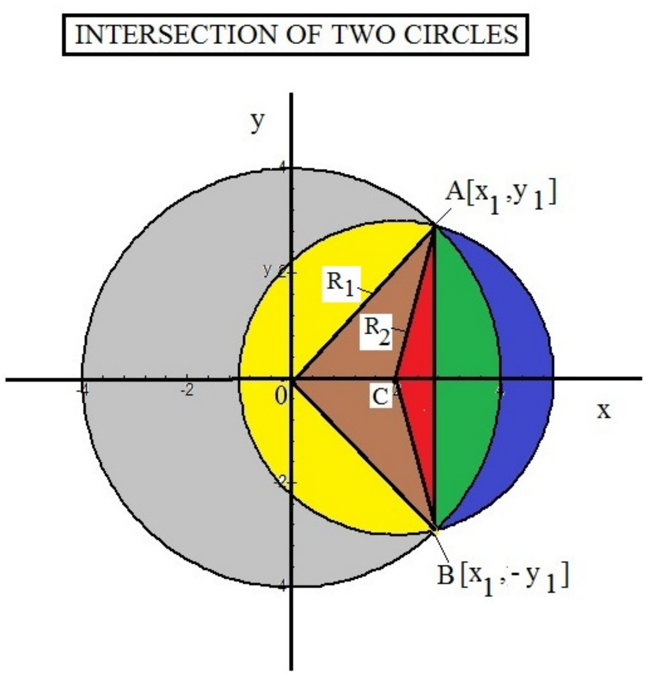
$$Area\ Segnent\ \ ADBC = R^2\arcsin(\frac{\sqrt{R^2 - C^2}}{R} - C\sqrt{R^2 - C^2}$$

These results completely describe the properties of lengths and areas formed by the intersection of a circle with a straight line. We have, for instance, that if R=2 and C=1, that the area of the sector AODB equals 4π/3.This equals 1/3rd of the total circle area. For this case, the segment shown in blue has an area 4π/3-sqrt(3).

We next examine the intersection of two circles-

$$x^2 + y^2 = R_1^{\ 2} \quad and \quad (x - C)^2 + y^2 = R_2^{\ 2}$$

This produces the following picture-



The intersection points are located at points A and B whose coordinates are–

$$x_1 = (\frac{1}{2C})\left\{R_1^{\ 2} - R_2^{\ 2} + C^2\right\} \quad and \quad y_1 = \pm\sqrt{R_1^{\ 2} - x_1^{\ 2}}$$

Notice that there are two distinct crescents shown in grey and blue in the above figure. Such crescents are also referred to as lunes. To get their areas we note that

$$Area(blue + green + red) = Area\ R_2\ Sector = R_2^{\ 2}\arcsin(\frac{y_1}{R_2})$$

$$Area(green + red + brown) = Area\ R_1\ Sector = R_1^{\ 2}\arcsin(\frac{y_1}{R_1})\ \mathrm{R}$$

and-

$$Area(brown) = y_1 C \quad , \quad Area(red) = y_1(x_1 - C)$$

Thus we can eliminate the areas of the various unwanted colors to get-

$$Area_{Small\ Crescent} = y_1 C + R_2^{\ 2}\arcsin(\frac{y_1}{R_2}) - R_1^{\ 2}\arcsin(\frac{y_1}{R_1})$$

$$Area_{Large\ Crescent} = y_1 C + R_1^{\ 2}\{\pi - \arcsin(\frac{y_1}{R_1})\} - R_2^{\ 2}\{\pi - \arcsin(\frac{y_1}{R_2})$$

Note that as the two radii become equal, both crescents have the same area of y_1C. The crescents will disappear when $C>R_1+R_2$. The lens shaped region between the two crescents takes on the shape of a football when $R_1=R_2$.This area is known in the mathematical literature as the Visica Piscis (fish bladder).

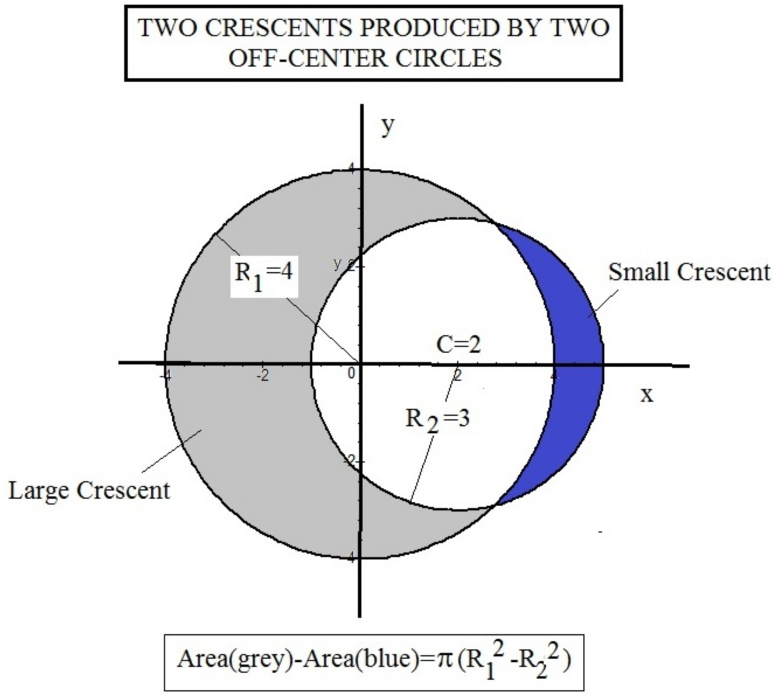
A student has posed a question on the internet- What is the area of the larger crescent formed by two circles of radii $R_1=4$ and $R_2=3$ with $C=2$. The answer is straight forward using the above area formulas. We have –

$$A_{Small\ Crescent} = \frac{3\sqrt{15}}{2} + 9\arcsin(\frac{\sqrt{15}}{4}) - 16\arcsin(\frac{3\sqrt{15}}{16}) = 4.66843...$$

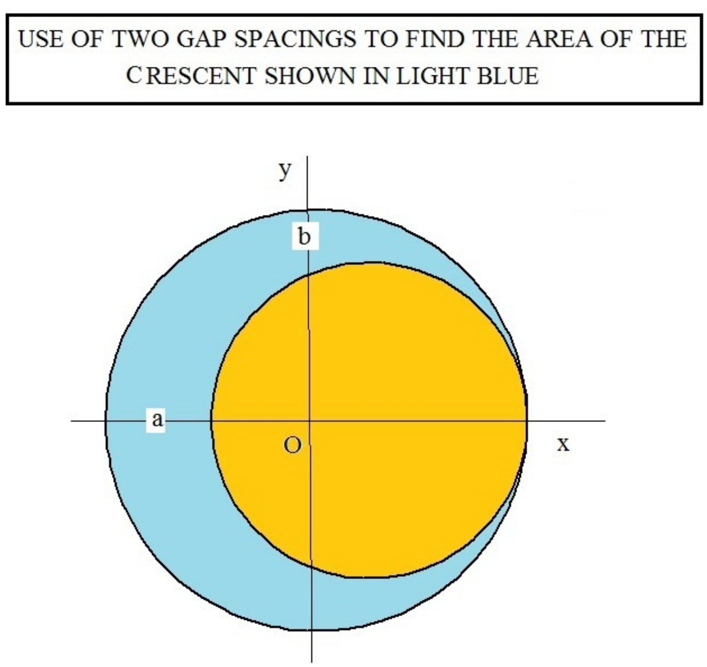
$$and$$

$$A_{Large\ Crescent} = \pi(16-9) + A_{SmallCrescent} = 26.65957...$$

Here is a picture of the configuration-



An even better crescent question posed on the internet involves a job application quiz in which the applicant is asked what is the area of the crescent formed by a large and small circle which touch each other at just one point along the positive x axis. Only the values of the gap widths ‘a’ and ’b’are given as shown-



I solve this problem as follows-

1)-Shift the smaller circle of unknown radius R_2 by distance a/2 to the left.
2)-We now have concentric circles with a constant gap of a/2.
3)-This means the area of the crescent must be $\pi(R_1^2-R_2^2)$ since no area is lost in the shift.
4)-Summing along the x axis we have $2R_1-a=2R_2$.
5)-From the right triangle formed by the shift, we have $R_2^2=(R_1-b)^2+(a/2)^2$.
6)-Solving for the radii we find-

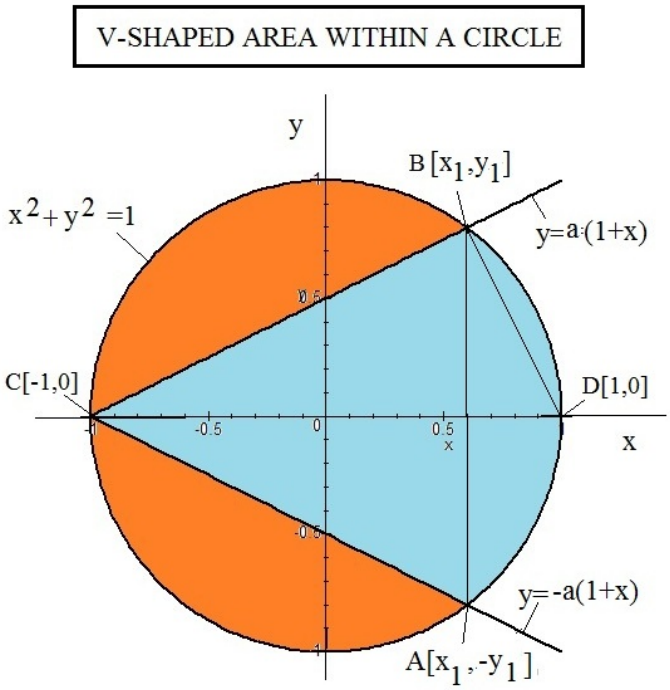
$$R_1 = \frac{b^2}{(2b-a)} \quad and \quad R_2 = \frac{b^2+(b-a)^2}{2(2b-a)}$$

7)-Thus the resultant area of the crescent is-

$$A_{crescent} = \frac{\pi}{\{2(2b-a)\}^2}\{4b^4 - [b^2+(b-a)^2\}^2$$

In the job application quiz the values of b and a were given as b=5cm and a=9cm. In that case the area of the crescent becomes 204.75 π.

We next see the type of areas which can be created by two lines and a circle. With appropriate scaling we have here the unit radius circle $x^2+y^2=1$ plus the lines y=a+bx and y=c+dx. Let us begin with the following configuration-



In this figure the two straight lines y=±a(1+x) both cross the x axis at x=-1 and intersect the unit radius circle at-

$$x_1 = \frac{(1-a^2)}{(1+a^2)} \quad and \quad y_1 = \frac{2a}{(1+a^2)}.$$

The resultant figure is reminiscent of the pac-man symbol in an early video game and also looks like the side-view of an an eye-ball. The blue V-shaped region has an area given by the sum of the isosceles triangle area ABC and the circle segment ADB. One finds-

$$Area_{V-Shape} = \arcsin(y_1) - y_1(1+x_1) = \frac{2a}{(1+a^2)} + \arcsin(\frac{2a}{1+a^2}) = 1.727\ldots$$

Each of the orange portions therefore has an area of –

$$Area_{Orange} = \frac{(\pi - Area_{V-Shape})}{2}$$

Another important result follows when one looks at the triangle CBD. Its three sides read-

$$sqrt\{(x_1+1)^2+y_1{}^2\} \quad , \quad sqrt\{(1-x_1)^2+y_1{}^2\} \quad and \quad 2$$

According to the Pythagorean Theorem this is a right triangle. Thus two lines drawn from opposite sides of a circle diameter and made to intercept on the circle, always produces a right angle between the lines. This is a result one learns quite early in high school trigonometry.

Another variation in the areas created with two straight lines and one circle follows from-

$$x^2+y^2=1 \quad , \quad y=ax \quad , \quad y=bx$$

These lines pass through the origin and intersect the circle at –

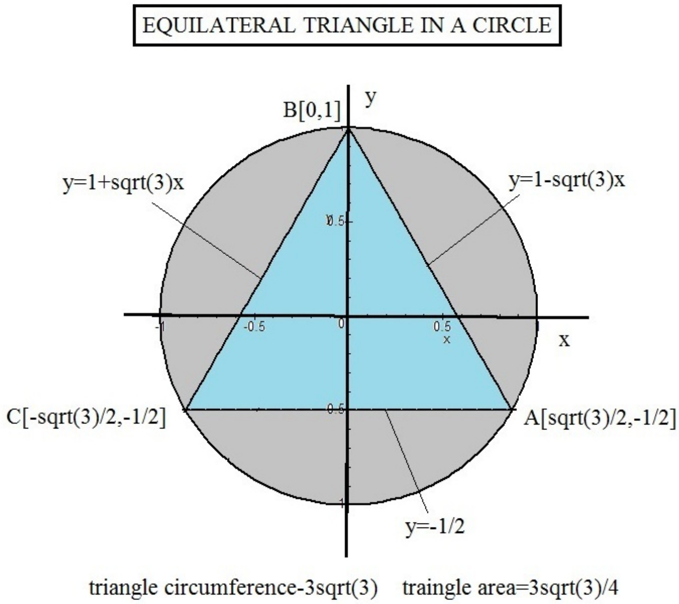
$$x_1=\frac{\pm 1}{\sqrt{1+a^2}} \quad , \quad y_1=\frac{\pm a}{\sqrt{1+a^2}} \quad , \quad x_2=\frac{\pm 1}{\sqrt{1+b^2}} \quad , \quad y_2=\frac{\pm b}{\sqrt{1+b^2}}$$

Using the dot product between the two lines, shows that the angle between the lines is-

$$\theta=\arccos\left\{\frac{(x_1x_2+y_1y_2)}{\sqrt{x_1{}^2+y_1{}^2}\sqrt{x_2{}^2+y_2{}^2}}\right\}$$

There are a total of two pair of sectors. Two of these have an area of θ each. The other two have an area of (π-θ) each.

We consider next three straigth lines and one circle. A figure using these elements is the following-



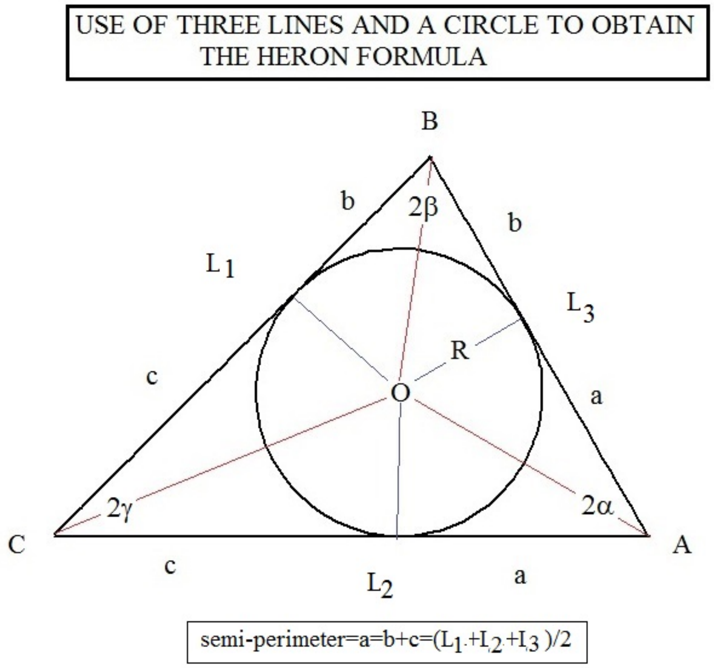
Here we have an equilateral triangle lying within a unit radius. The vertices lie on the circle and are separated by 120deg from each other. Each side of the triangle has length sqrt(3). The area of the triangle is-

$$A_{Triangle} = \frac{3\sqrt{3}}{4} = 1.2990\ldots$$

This means each of the three segments of the circle shown have an area of-

$$A_{Segment} \frac{(\pi - \frac{3\sqrt{3}}{4})}{3} = \frac{\pi}{3} - \frac{\sqrt{3}}{4} = 0.61418...$$

We point out that any triangle can be either inscribed in or circumscribed about a circle such that its vertices sit on the circle. Here is an example of three lines used to form a triangle which circumscribes a circle-



This configuration is historically important one since it can be used to find the area of any triangle in terms of its semi-perimeter as done by Hero of Alexandria in about 60AD. Let us show how this is accomplished. First one draws three lines bisecting the triangle vertices. Next one draws three lines perpendicular to the sides and passing trough the circle center. This produces a total of three right triangle pairs whose area must equal the area of the large triangle. We thus have-

$$A_{BigTriangle} = (a+b+c)R = \mathrm{sR}$$

Here s is the semi-perimeter a+b+c. Next one seeks a way to write R in terms of a,b, and c. Using the Law of Sines we have-

$$\frac{\sin(2\alpha)}{(b+c)} = \frac{\sin(2\beta)}{(a+c)} = \frac{\sin(2\gamma)}{(a+b)}$$

But we also know from trigonometry that-

$$\sin(2\theta) = \frac{\tan(2\theta)}{\sqrt{1+\tan(2\theta)^2}} = \frac{2\tan(\theta)}{1+\tan(\theta)^2}$$

Also, since tan(α)=R/a, tan(β)=R/b, and tan(γ)=R/c, we can write the Law of Sines as-

$$\frac{a}{(R^2+a^2)(s-a)}=\frac{b}{(R^2+b^2)(s-b)}=\frac{c}{(R^2+c^2)(s-c)}$$

Cross multiplying the first two equations produces-

$$R=\sqrt{\frac{abc}{s}}$$

From this we have Heron's Triangle Formula-

$$Area_{ABC}=Rs=\sqrt{sabc}=\sqrt{s(s-L_1)(s-L_2)(s-L_3)}$$

Notice that we were able to derive this result without using the elaborate geometrical construction used by Heron in his original derivation. According to this formula, the area of an equilateral triangle of sides L each has area-

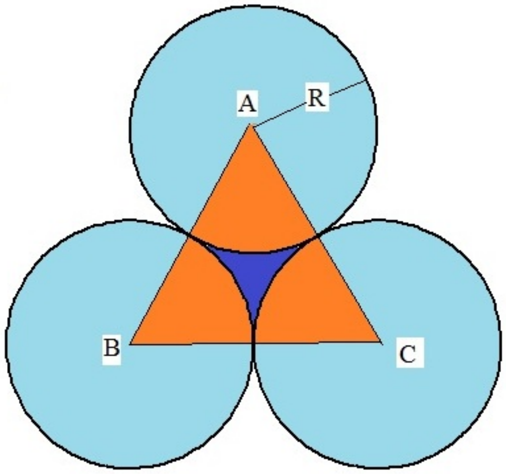
$$Area_{EquiTriangle}=\sqrt{(\frac{3L}{2})(\frac{3L}{2}-L)^3}=\frac{\sqrt{3}}{4}L^2$$

We conclude our discussions by looking at the case of the area created by three equal size circles centered at the vertices of an equilateral triangle and just touching each other. One possible set of equations for these circles is-

$$x^2+(y-1)^2=(3/4)$$
$$(x-\sqrt{3}/2)^2+(y+1/2)^2=(3/4)$$
$$(x+\sqrt{3}/2)^2+(y+2/2)^2=(3/4)$$

They produce the graph-

THREE TOUCHING CIRCLES



The new area of interest here is the small dark blue triangle like region. Its three sides form part of the circles. One recognizes that its area just equals the area of the equilateral triangle shown minus the area of the sixty degree sectors of the circles shown in orange. We find that the blue scalloped region has area-

$$A_{Blue} = \{\sqrt{3} - \frac{\pi}{2}\}R^2 = 0.16125R^2$$

This area is quite small compared to the area of a circle of area πR^2. We have found this fact quite useful in some of our earlier work on heat transfer along a bundle of open ended capillary tubes.