CALCULATING THE VALUES OF THE SIX MOST IMPORTANT MATHEMATICAL COSTANTS USING SEVERAL NEW APROACHES

Mathematics contains many constants of an irrational nature. The most important of these are –

$$\pi = 3.1459.. , e = 2.718.. , \sqrt{2} = 1.414.., \ln(2) = 0.6931..$$
$$, \varphi = 1.618..., and \quad \gamma = 0.5772...$$

We want here to calculate the value of these numbers using some new approaches not relying on the standard infinite series summations which in most cases are slowly convergent.

SQUARE ROOT OF TWO:

Let us begin with the root of two. Here we first write-

$$(\sqrt{2}-1)(\sqrt{2}+1) = 1$$

Expanding this equality as a continued fraction we get-

$$\sqrt{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1+\sqrt{2}}}}}$$

We next iterate this result by replacing the left term by S[n+1] and the right by{A(k)+B(k)S[n]}/{C[k]+D[k]S[n] , with k being the number of terms taken in the continued fraction. Doing this we get the following table-

k	1	2	3	4	5	6	7	8	9	10
A{k)	2	4	10	24	58	140	338	816	1970	4756
B(k)	1	3	7	17	41	99	239	577	1393	3363
C(k)	1	3	7	17	41	99	239	577	1393	3363

D(k)	1	2	5	12	29	70	169	408	985	2378

The larger k is taken the faster the iteration will converge to sqrt(2). You will note that B(k)=C(k), A(k)=2D(k), and B(k+1)=A(k)+B(k)+C(k) . Taking k=10 we get the rapidly convergent iteration formula-

$$S[n+1]=\frac{4756+3363S[n]}{3363+2378S[n]}$$

Starting with S[1]=1.5 we get the following results-

FIRST THIRTEEN ITERATIONS FOR SQRT(2) USING S[n+1==(4756+3363S[n])/(3363+2378S[n])

S_1=1.5
S_2 := 1.41421356
S_3 := 1.4142135623730950
S_4 := 1.414213562373095048880168
S_5 := 1.41421356237309504880168872 4209
S_6 := 1.4142135623730950488016887242096980785
S_7 := 1.414213562373095048801688724209698078569671875 3
S_8 := 1.41421356237309504880168872420969807856967187537694 8073
S_9 := 1.4142135623730950488016887242096980785696718753769480731766797
S_{10} := 1.414213562373095048801688724209698078569671875376948073176679737990732
S_{11} := 1.41421356237309504880168872420969807856967187537694807317667973799073247846210
S_{12} := 1.41421356237309504880168872420969807856967187537694807317667973799073247846210703885 0
S_{13} := 1.4142135623730950488016887242096980785696718753769480731766797379907324784621070388503875343 2
S_{14} := 1.4142135623730950488016887242096980785696718753769480731766797379907324784621070388503875343276415 73

100 digit accuracy at S[14] using S[1]=3/2

In these calculations we have terminated the values of S[n] where they first depart form sqrt(2). It is seen that for S[14] we get the hundred digit accurate result-

$$\sqrt{2}=1.4142135623730950488016887242096980785696718753769480731766797379907324784621070388503875343276415 73$$

GOLDEN RATIO φ:

After the root of two, the golden ratio, thought by the ancient Greeks as the ideal ratio of height to width of a picture frame, is the easiest to calculate numerically to any order of accuracy. Its mathematical definition is-

$$\varphi = \frac{1+\sqrt{5}}{2}$$

So we need only determine the root of five to get its value. Root five satisfies the identity-

$$(\sqrt{5}-2)(\sqrt{5}+2=1$$

This result may be expressed as the continued fraction-

$$\sqrt{5} = 2 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\sqrt{5}}}}$$

This allows us to set up the iteration formula-

$$S[n+1] = \frac{A(k)+B(k)S[n]}{C(k)+D(k)S[n]}$$

starting with S[1]=2. The values of the constants A through D are given in the following table-

k	1	2	3	4	5	6	7
A(k)	5	20	85	360	1525	6460	27365
B(k)	2	9	38	161	682	2889	12238
C(k)	2	9	38	161	682	2889	12238
D(k)	1	4	17	72	305	1292	5473

Note here B(k)=C(k), A(k)=5D(k), and B(k+1)=A(k)+2B(k). Using the iteration formula corresponding to k=7 we find –

S[13] :=
2.236067977499789696409173668731276235440618359611525724270897245410520925637804899414414408378782275

This corresponds precisely with the first 100 digits of sqrt(5). Using this result we get the golden ratio to equal-

$$\varphi \approx \frac{1+S[13]}{5} = 1.61803398874989484820458683436563811772030917980576286213544862270526046281890244970720720418939113 8$$

This also has one hundred digit accuracy.

EXPONENTIAL EXP(1):

This irrational number represents the base for natural logarithms with the property that the derivative of exp(x) equals to itself. One usually writes exp(1) as the letter e. Its evaluation is pretty rapid even when using the infinite series representation-

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

The factorial makes the convergence quite rapid. To get a hundred digit accuracy one needs to sum this series up to $n!=10^{100}$. This means the first seventy terms. An alternate way which is much simpler for finding exp(1) is to note that the integral-

$$K[n] = \int_{x=1}^{1} P_{2n}(x)\cosh(x/2)dx = \exp(1/2)M(n) - \exp(-1/2)N(n)$$

Here $P_{2n}x$ are the even Legendre polynomials which oscillate rapidly as n gets large while the hyperbolic cosine varies rather slowly in 0<x<1. The N(n) and M(n)) are polynomials in n. As n gets large the integral K(n) approaches zero leaving one with the approximation-

$$\exp(1)=e\approx N(n)/M(n)$$

For n=15 we find-

$$\exp(1) \approx \frac{51610959626630564395418271773697752132736915770371}{18986610985766723481463367409286454253253111186111}$$

$$= 2.71828182845904523536028747135266249775724709369\ 9959574\ 96696762772407663035354759457138217852516642 7$$

which is accurate to 100 decimal places. The reason this approach is so accurate stems from the fact that the oscillatory nature of the Legendre polynomials guarantees that the integral K(n) is essentially zero when n gets large.

VALUE OF π:

Here again we make use of even Legendre polynomials and note that -

$$K[n] = \int_{x=0}^{1} \frac{P_{2n}(x)}{1+x^2} = M[n]\pi + N[n]$$

To get a 100 digit accuracy one must go to n=62. On setting K[62] to zero, one finds-

$$\pi = 3.1415926535897932384626433832795028841971693993751058\\ 2097494459230781640628620899862803482534211706\,8$$

To reduce the number n for the Legendre polynomials one needs to increase the term 1 in the denominator of the integral. Replacing 1 by 3 allows one to reduce n to 50 for the same 100 digit accuracy. Also one can use the sum of multiple arctan series in conjunction with Legendre polynomials to get the same accuracy with a further reduction in n. Several decades ago we came up with the interesting identity-

$$\pi = 48\arctan(\frac{1}{38}) + 80\arctan(\frac{1}{57}) + 28\arctan(\frac{1}{239}) + 96\arctan(\frac{1}{268})$$

, which has all positive terms. The denominators in the arctan functions are all large integers. Using the Legendre polynomial approach on this last equation for n=26 produces the same one-hundred digit accurate result for π.

NATURAL LOGARITHM OF TWO:

Another of the six most encountered mathematical constants is the natural logarithm of two. We can define this number as-

$$\ln(2) = \int_{x=0.}^{1} \frac{dx}{(1+x)} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - ...$$

The series expansion to the right of the integral is also known as the Gregory series . It is notoriously slow in converging although its alternating character allows one to state that –

$$0.5<\ln(2)<0.8333..$$

To improve the convergence rate we first look at the series-

$$\ln(1+x)=\mathrm{x}-\frac{\mathrm{x}^2}{2}+\frac{\mathrm{x}^3}{3}-\frac{\mathrm{x}^4}{4}+0(\mathrm{x}^5)$$

This series converges most rapidly as x approches zero So if we want to find ln(2) we need to use an identity of the form-

$$2=1+x=AB/CD$$

making sure A, B, C,and D remain near one each. One possibility is A=B=1.2 and C=0.8 with D=0.9. This yields the identity-

$$\ln(2)=2\ln(1.2)-\ln(0.8)-\ln(0.9)$$

We can now estimate the values of the three logarithms by looking at-

$$K(n,a)=\int_{x=0}^{1}\frac{P_n(x)}{[(1+\frac{1}{a})+x]}dx$$

for a=5,-5, and -10. For 100 digit accuracy we need to choose n=96. This produces the result-

$$\ln(2)=.6931471805599453094172321214581765680755001343\\6025525412068000949339362196969471560586332699641868 9$$

EULER-MACHERONI CONSTANT γ :

This constant was studied in detail by Leonard Euler(1734) and later by Lorenzo Macheroni.It represents essentially the area between, the staircase function S(n)=1/n and ln(n) as n goes to infinity. Mathematically the definition reads-

$$\gamma(n)=\frac{\lim}{n\to\infty}[\sum_{k=1}^{n}\frac{1}{k}-\ln(n)]$$

where $\gamma(\infty)$ means the Euler Constant γ. The function γ (n) converges very slowly toward γ and so is not used for finding γ to high accuracy. However there is an alternate integral approach which quickly produces the value of γ to high accuracy. The procedure for doiung this is to first define the Psi(x) function-

$$\text{Psi(x)}=\Gamma(x)'/\Gamma(x)== \int_{t=0}^{\infty} \ln(t)\exp(x-1)\exp(-t)dt$$

Then setting x=0 one obtains-

$$\text{Psi(1)}= \int_{t=0}^{\infty} \ln(t)\exp(-t)dt = Laplace[\ln(t) \quad with \quad s=1]=-\gamma$$

On substituting exp(-t)=u, we get the definite integral-

$$\gamma = -\int_{u=0}^{1} \ln\{-\ln(u)\}du$$

This integral needs only be evaluated over the finite range 0<u<1 and can be evaluated to any order of accuracy. To 100 places we get-

γ=0.5772156649015328606065120900824024310421593359399235988057672348848677267776646709369470632917467492

SUMMARY:

We have found the values to one hundred places for the six most important irrational constants arising in mathematics. The approach used is an unconventional one involving an iteration approach where the S[n+1] iteration is expressible as the linear fractional form (A+BS[n])/(C+dS[n]) and values of the remaining numbers are determined by using integrands of rapidly oscillating Legendre polynomials and slower varying functions of x. These techniques should continue to hold for any order of accuracy.

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