WHAT IS MENTAL ARITHMETIC?

By mental arithmetic one means carrying out the basic mathematical operations of addition, subtraction, multiplication and division in one's head without the aid of pencil and paper, tables, hand calculators, mechnical or electronic computers. It is a skill encouraged among elementary and middle school pupils when they are first instructed on the basic mathematical operations available for integers. After ninth grade such mental arithmetic operations are unfortunately no longer emphasized (since hand calculators are used to replace things). Nevertheless, these mental operations will often come in handy on many occasions in later life such as when quickly determining restaurant tips or bank interests. We want here to discuss the basic approaches used in mental arithmetic.

ADDITION:

This is the easiest of the four basic mathematical operations. Here one starts with the summing of two numbers, say,-

2789+342

In ones's head it at once can be written as 3000+131=3131. It is worked out by starting on the left and going to the right. It should be valid for any sum of integers of specified digit length. Sometimes partial sums can be obtained early in the calculations. Thus

871+109+41=980+41=1021 or 3482+1276=4600+158=4758

Sometimes some interesting results can be obtained involving the sum of a finite number n of integers related to each other. Take, for example,-

1+2+3+4+5+...+n

This is equivalent to the two term groupings n+1 taken n/2 times. Thus we find the sum–

n(n+1)/2

Consider next the infinite series-

1+1/2+1/4+1/8+1/16 +1/32+….

Grouping we get 3/2+3/8+3/32+3/128+…=$(3\cdot 5\}/(2\cdot 4)\{1+\frac{1}{16}+\cdots\}\rightarrow 2$

SUBTRACTION:

This operation is treated the same as addition except that we also include negative integers. Take-

467-322=100+45=145 and 7523-4674=3000-151=2849

We can also deal with mixed signs such as-

476-321+21-781=-305-300=-605

All of the above manipulations can be done by mental arithmetic after a little practice.

MULTIPLICATION:

This operation is a bit more difficult than addition and subtraction, but it can be carried out with a little practice. Take-

57*32=1920-96=1824

The square of a number becomes 39^2=(40*39)-39=1560-39=1521. Take next-

256*32=2*2^12=8192

In the last case we made use of the powers of two. One can always simplify an expression by factoring out the common denominators. Thus-

560*350=560*70*5=40*700/2=196000

Note there will usually be more than one way to reach an answer. One typically chooses the simplest of these possible routes.

DIVISION:

Consider the division 2548/13. We note that 2548=4*49*13, so that 2548/13=196. Often the division will not be exact so that a fraction remains. An example of this is-

119/37=3+2/7=4-5/7 =3.2857…

There are examples in history where individuals have the specific ability for mental calculations involving large numbers throughout their lives leaving observers baffled by demonstrations. One of these individuals was A.C.Aitken(1895-1967) of the University of Edinburgh, Scotland. He was able to mentally handle combined summations and products involving numbers of hundred digit length. Certain mathematical savants also have this ability.

PRODUCTS AND ROOTS OF NUMBERS:

The simplest way to calculate a square is to write as follows-

$$38^2=(40-2)*(36+2)=1440+8-4=1444$$

or-

$$(123)^2=(120+3)*(126-3)=15120+18-9=15129$$

To find the fifth power of seven we have-

$$7^5=49*343=17150-343=16807$$

Also when finding the square S[n] of a number n one can use the identities-

$$S[n+1]=S[n]+2n+1 \quad \text{and} \quad S\{n]=S[n-1]+2n-1$$

This allows quick calculation of the squares S[n+1 or S[n-1] when S[n] is known. Thus the square of 26 equals 625+51=676. The square of 19 will be 400-39=361.To become proficient at mental math it is a good idea to recognize the squares of numbers through about 50 and the exponents of 2 through about 2^10=1024.

The square root of the number 2809 is–

$$sqrt(2809)=53$$

The standard way to work this out mentally is to note that the ending 9 comes from 3 or 7 squared. Ignoring 09 leaves us with 5<sqrt(28). So the root is either 53 or 57. The answer is 53

Consider next sqrt(4761). The 1 ending comes from 1 or 9.The root of 47<7^2. So we have a root of either 61 or 69. It is the second one which yields sqrt(4761)=69. You can check your result by evaluating 69*70-69=4761. This method breaks down for the roots of numbers above six digits since the root of a 4 digit number or higher is not always known.

An alternative more general way to take a root is by series expansion as follows-

$$sqrt(4761)\approx 70[1-139/7200]=69.00714$$

So the answer must be sqrt(4761)=69. The idea here is to pick the nearest perfect square N=4900.

SOME OTHER MENTAL CACULATIONS:

As a teenager one usually learns about compound interest and tip percentages from ones parents or peers. The basic formula for compound interest is –

$$C/Co=(1+i)^n$$

Here C/Co is increase ratio of invested capital after n years. The interest rate is set as i. Taking the natural logarithem of both sides and noting that i<<1, we have-

$$\ln(C/C_o)=ni\{1-i/2+i^2/3-\ldots\}$$

Retaining only the first term in the curly bracket yields the approximation-

$$ni\approx\ln(C/Co)$$

Many refer to this last result as the rule of 70 since it says that 7% interest will produce a doubling of one's capital C in a decade. Recall that ln(2)=0.693…

Another practical problem is to quickly determine a 15% tip on a $125.76 restaurant bill. Without needing a pocket calculator , one performs the operation 377.28/20 to get $18.86.

Finally, let us calculate the miles per gallon of an automobile going 347 miles on 19 gallons of gasoline. The mpg=347/19=18+5/19=18.263.

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