ON THE NUMBER OF PRIMES CONTAINED IN THE 
FIRST N INTEGERS

INTRODUCTION:

It is well known since the time of Gauss and Legendre that the number of prime numbers found in the first N integers goes as N/\ln(N) provided that N is very large. This observation is known as the Prime Number Theorem and usually appears in the literature as-

\[ \pi(N) \approx \frac{N}{\ln(N)} \]

When N is smaller this result tends to under estimate the actual number of primes by an amount which decreases toward zero as N approaches infinity. As an example, the number of primes in the first 100 integers is exactly 25 while the Prime Number Theorem predicts the lower value of 100/\ln(100)=21.7. For N=1000 the Prime Number Theorem predicts 144.8 primes while in reality there are 168 primes.

It is our purpose here to determine the exact prime number fraction for Ns in the range 20<N<10,000 and show how the Prime Number Theorem can be modified to give a good estimate for the number of primes over this range.

DETERMINING THE EXACT PRIME FRACTION F(N)

We start with the following table-

<table>
<thead>
<tr>
<th>Integer N</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primes P</td>
<td></td>
<td>2</td>
<td>3</td>
<td>5</td>
<td></td>
<td>7</td>
<td></td>
<td>11</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Prime Fraction</td>
<td>1</td>
<td>2/3</td>
<td>2/4</td>
<td>3/5</td>
<td>3/6</td>
<td>4/7</td>
<td>4/8</td>
<td>4/9</td>
<td>4/10</td>
<td>5/11</td>
<td></td>
</tr>
</tbody>
</table>

The first row gives the integers in ascending order, the second row represents the prime numbers up to and including N, while the third row yields the prime fraction F(N) in quotient form. We define this prime fraction as-

\[ F(N) = \frac{\text{NUMBER OF PRIMES}}{N} = \frac{n}{\text{ithprime}(n)} = \frac{n}{N} \quad \text{with} \quad n < N \]

, provided that N=ithprime(n). Here n represents the number of primes less or equal to N. So from the above table we get [n,m]=[1,2],[2,3],[3,5],[4,7],[5/11]
We have used our MAPLE computer program to obtain a graph of N versus F(N) in the range 3<N<10000. Here are the exact results shown in red-

On the graph we have also placed the Prime Number Theorem result which for the prime fraction reads 1/ln(N). Note that in this range of N the 1/ln(N) curve lies below the exact solution curve by about 10%. It is only as N becomes extremely large that the theorem holds as recognized quite early by Legendre and Gauss who established the theorem by brute force divisions plus some extrapolation. This was an amazing task in those pre-electronic computer days. Gauss took his calculations out to about N=200000 mentioning that he was never able to reach a million. These days it is an easy task to find both F(N) and 1/ln(N) for specific values of N=ithprime(n). Here is a short table-

<table>
<thead>
<tr>
<th>n</th>
<th>N</th>
<th>F(N)=n/m</th>
<th>1/ln(N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>11</td>
<td>0.45454545</td>
<td>0.41703239</td>
</tr>
<tr>
<td>25</td>
<td>97</td>
<td>0.25773195</td>
<td>0.21859304</td>
</tr>
<tr>
<td>168</td>
<td>997</td>
<td>0.16850551</td>
<td>0.14482781</td>
</tr>
<tr>
<td>1229</td>
<td>9973</td>
<td>0.12323272</td>
<td>0.10860550</td>
</tr>
<tr>
<td>9592</td>
<td>99991</td>
<td>0.09592863</td>
<td>0.08685957</td>
</tr>
<tr>
<td>78498</td>
<td>999983</td>
<td>0.07849933</td>
<td>0.07238250</td>
</tr>
</tbody>
</table>

Note, as expected, N> n and F(N)>1/ln(N).
FORMULA FOR PREDICTING THE PRIME FRACTION MORE ACCURATELY IN 20<N<10000:

To modify the Prime Number Theorem for N<10000 we use an approach first suggested by Legendre in his book “Essay on the Theory of Numbers”. In it he states that for smaller Ns, the prime number Theorem should read-

\[ \pi(N) \approx \frac{N}{A \ln(N) + B} \]

, where A and B are to be determined. Let us try this approach. First of all we already know from Gauss that \( \pi(N) \) should approach \( \frac{N}{\ln(N)} \) as \( N \) goes to infinity. This means that \( A=1 \). It leaves us with the equality-

\[ F(N) = \frac{1}{B + \ln(N)} \]

so that we get –

\[ B = \frac{1}{F(N)} - \ln(N) \]

, which will be negative. From it also follows-

\[ \pi(N) = \frac{N}{\ln(N) + \frac{1}{F(N)} - \ln(N)} = NF(N) \]

If we now pick a small N such as \( N=97 \) where \( n=25 \), we have \( F(N)=n/N=25/97=0.4545454 \). So B becomes –

\[ B = \frac{97}{25} - \ln(97) = -0.6947 \]

To the nearest tenth this allows us to write a Legendre type of modified Prime Number Theorem-

\[ \pi(N) = \frac{N}{\ln(N) - 0.7} \]

A comparison of the denominator of this function with \( F(N) \) in the range 20<N<10,000 follows-
The comparison of \( \pi(N)/N \) with \( F(N) \) is quite good down to about \( N=20 \). Below this the approximation departs to the upside.

**NUMBER OF PRIMES IN A GIVEN INTERVAL OF N:**

With the new approximation for \( F(N) \) we are now in a position to determine the number of primes in a given interval of \( N \). Let us ask what is the number of primes we can expect between \( N=100 \) to \( N=200 \). Using the un-modified Prime Number Theorem we find this number to be-

\[
\frac{200}{\ln(200)} - \frac{100}{\ln(100)} = 37.75 - 21.71 = 16.04
\]

The modified form yields-

\[
\frac{200}{[\ln(200) - 0.7]} - \frac{100}{[\ln(100) - 0.7]} = 43.49 - 25.61 = 17.88
\]

If we now check the ithprime terms stored in our math program between \( n=100 \) and \( N=200 \), we find exactly 21 primes. This exact result differs from the prime number Theorem and its modification by 24% and 15%, respectively.
From the above table we know that N=99991 has n=9592 so that F(N)=n/N=0.0959286. That is the percent of the numbers below 99991 which are prime is 9.59%. The estimation by the modified formula yields-

$$\frac{1}{\ln(9991) - 0.7} = 0.0924826$$

That is, the estimation formula predicts that 9.25% of the numbers below N=9991 are primes. This is not a bad estimate.

**CONCLUDING REMARKS:**

One can modify the Prime Number Theorem to give a reasonable estimate for the number of primes expected for any N>20. At very large N this approximation merges into the 1/ln(N) result for the fraction of primes lying below N. Also one can use the result to get good estimated of the number of primes expected in the range N_1<N<N_2. We estimate that the number of primes n lying below N equal to one million equals 7.624%. That is one estimates n=76246. The actual number is n=78498.

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Feb.7, 2018  
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