MORE ON MENTAL ARITHMETIC

The earlier introduction of the New Math in grades K through 12 and the presently used Common Core have had the effect of making our grade school students less proficient in the four basic mathematical operations of adding, subtracting, multiplying and dividing. These days one often sees individuals at restaurants who have absolutely no idea how to calculate a 17% tip on a $37.78 dinner bill without resorting to a hand calculator (or their I-Phone or new Apple watch) . Also I have often found that store clerks who are products of this type of modern math are unable to subtract the difference between a charge and what money a customer lays on the counter without first using their register to calculate the expected change. They are unable to do this mentally. I attribute these facts to the lack of emphasis in public schools on the four simple mathematical manipulations mentioned above. Very few students will make use of the admittedly important and more advanced mathematical concepts of sets, matrices, modular arithmetic, vector spaces and number theory in later life, but they will need to be able to effectively carry out the mentioned elementary math manipulations and do so rapidly and accurately preferably mentally. It is our purpose here to show how with a little practice anyone can carry out the basic mathematical operations of addition, subtraction, multiplication and division accurately without the aid of pencil and paper or computer and, at the same time, have an understanding of what they are doing and not just applying an algorithm.

ADDITION:

Let us begin with addition. This is the easiest of the four basic math operations. In the classical way one sums the following four 3 digit numbers as follows-

```
     743
     147
     256
 +   874
-----------------------
    2020
```

First you add up the four numbers in the left column to get 20. You mark down the 0 and carry 2 to the second column. Adding the elements in the second column plus the carry yields 22. So mark down a 2 to the left of 0 and carry a 2 into the third column. Adding this third column including the carry leaves one with 20. Adding this to the left of 20 produces the result 2020. A student will not necessarily know why this algorithm works but he can always get a quick answer needing to know only his addition table for single digit numbers.

A more sophisticated way to get the same result is to recognize that the sum of the elements in the left column represent 18 hundreds, the sum of the integers in the second column equal 20 tens, and the sum of the elements in the third column are 20 ones. So adding these together we have-

```
 1800
  200
   20
---------------
  2020
```

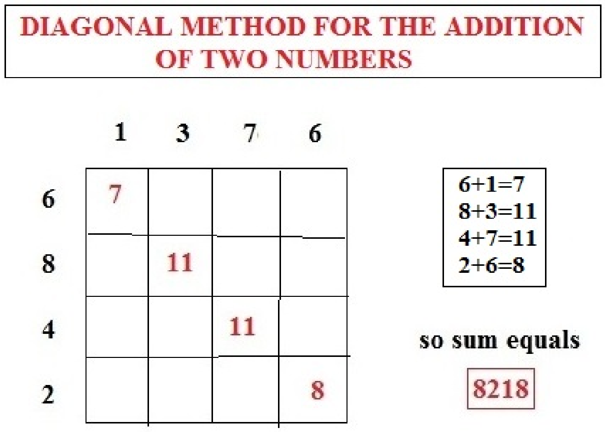
The same result, but obtained with a much better understanding of what decimal places mean. This second way of adding is identical to what abacus users employ. One is also free to add a number from one row and subtract the same amount from another. Thus in the above we could write-

```
 2000              2020
    0      →          0
   20                 0
---------------    -----------
 2020              2020
```

For the actual summation of numbers one need not go through these last few manipulations but simply rely on the number of hundreds, tens and ones in the sum. Here is how I would handle the summation of-

```
   34578           190000
   76120    →       13000
   82345             1700
+  11854              180
                       17
                 ----------------
                   204897
```

There is really nothing more to adding then to sum up the single digit integers in a given column and then placing these sums at the correct decimal point location. With a little practice these manipulations can be done in one’s mind without resorting to calculators or pencil and paper. Try it for some cases of your own. Notice there is no need to carry any numbers in our vertical sums. I often find it convenient in adding sums in my mind to derive the numbers in the diagonal in an n x n array and then read off the answer directly. Let me demonstrate by adding 1376 to 6842. Here we have the arrangement-



The elements along the diagonal are just the sums of the corresponding single digits in the numbers being added. Such a pictorial representation for the addition of two numbers looses its simplicity when more than two numbers are involved.

SUBTRACTION

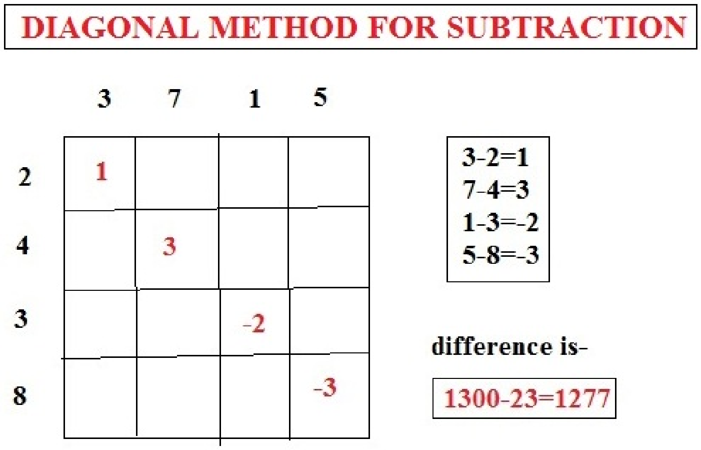
This is the reverse of addition meaning N=-(-N), where N is any number positive or negative. So by the decimal approach-

983 - 651 =300+30+2=332

, with confirmation given by 651+332=983. So division just requires a thorough knowledge of the subtraction table for single digit numbers. Here is a more complicated case-

71236-42181=30000-1000+100-50+5=29055

Again there was no need to carry any numbers. A pictorial representation of the difference of two numbers follows-



The elements along the diagonal are gotten by subtracting corresponding digits from each other. Here we get 3715-2438=1277. If the number of digits in the two numbers differ than one fills in the spaces in front of the smaller number by zeros. Thus 24678-3569=24678-03568=2,1,1,1,0= 2 1 1 1 0 . You can also combine addition and subtraction in the same problem. Thus-

```
  46713          90000         94100
- 06782   →       4000   →              →     93985
- 24596            100          -115
  78650          - 110
------------       - 5
```

Decimal fractions are also easily to handle. We have 6/16 +7/8-3/64=77/64=1.203125.

MULTIPLICATION

This operation is a bit more involved than simple addition and subtraction. The classical way to find the product of two numbers goes as follows-

```
      345
    x 691
 ----------------
      345
    3105
 +  2070
-----------------
   238395
```

This method works fine but is a bit slow and does involve carrying numbers. A faster way is to recognize that this product equals 345 x (700-9)=241500-3105=238395. or even better you can write-

345 x 691=(300+45) x (700-9)=210000+31500-2700-405=238395

The last result gives one the clue that the product of two 2 digit numbers breaks up into the following form-

77 x54=(80-3) x(50+4)=40 hundreds+(32-15) tens+(-12)ones=4000+170-12=4158

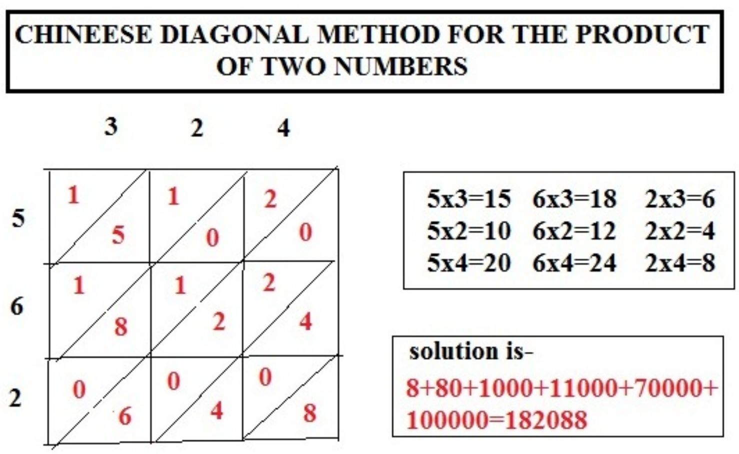
Casting this into generic form says-

(ab) x(cd)=100ac+10(ad+bd)+bd

So we see at once that 37^2=900+10(42)+49=1369 and 99^2=8100+1620+81=9801. Also we have-

34 x 72=21hundreds+10(28+6)tens+8ones=2448

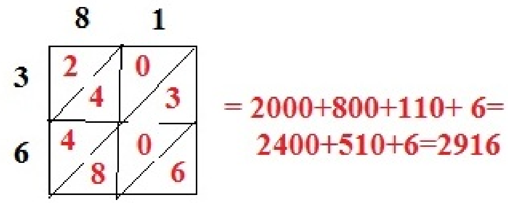
There is an ancient Chinese way to pictorially indicate how the product of two N digit numbers can be found. It goes as follows-



As one sees it involves the product of corresponding single digit elements in the two numbers. These are places into the nine squares such that the one digit part lies in the lower and the two digit part in the upper half of the square. The product is then determined by adding up the elements between the diagonals. What this algorithm is essentially equivalent to is a product expansion. Let us demonstrate this for the simpler case of-

36 x81=2400+10(51)+6=2916

The Chinese diagonal summing version of this is-



So the two different calculating methods are essentially identical.

To evaluate the product of two three digit numbers one must write things as follows-

$$abc \equiv 100a+10b+c \qquad def \equiv 100d+10e+f$$

On multiplying the two numbers together we get the product-

$$10000ad+1000(ae+db)+100(be+af+cd)+10(ec+bf)+cf$$

If now a=1,b=2,c=3 ,d=4,e=5, and f=6, we get-

$$123 \times 456= 40{,}000+13{,}000+2{,}800+270+18=56088$$

This works but is rather lengthy and less likely to be able to be done from memory as we are able to do with two digit number products. However, it is always possible to reduce A n digit number down to recognizable terms. Let us demonstrate this with-

$$237 \times 469=200(469)+37 \times 400+37 \times 69=93800+14800+1800+10(69)+63=111153$$

DIVISION

This is the last and most difficult of the fundamental four arithmetic operations. It is the inverse of multiplication and follows the form-

$$A/B=C \quad \text{with } C \times B=A$$

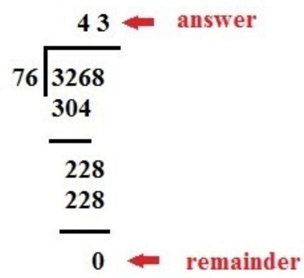
The classical way of dividing two numbers

$$3268 \div 76 = 3268/76$$

Goes as follows-

CLASSICAL PROCEDURE FOR DIVIDING TWO NUMBERS

Numerator=3268 **Denominator=76**



This algorithm takes the smaller denominator and divides it into 326 to get 4 and a remainder of 22<76. Next we bring down the 8 and divide 76 into 228. This goes exactly three times and leaves no remainder. So the answer is 43. To check the result one can multiply 43 x 76 to get 3268. The essence of the method is to find largest integer which divides the denominator into the numerator. Next carry out the similar procedure on the positive remainder. The procedure is repeated until the final remainder is zero or one stops after any desired number of digits have been found.

It is of course not necessary to write down all the steps shown in the above example. Much, if not all , of which can be done in one's head. Let me demonstrate by choosing a denominator of 34 and a numerator of 826. We see at once that 34 goes twice into the 82 with a remainder of 14 left over. Next 146 divided by 34 produces 4 with 10 remainder. Then 100/34=2 and 32 remainder. Then 320/34=9 plus 14 remainder. Finally 140/34=4 and 4 remainder. Adding things up we get-

$$826 \div 34 = 24.294\ldots$$

Symbolically we memorize things as follows-

$$2 \to (146) \to 4 \to (100) \to 2 \to (320) \to 9 \to (140) \to 4 \to (40)$$

Here the un-bracketed single integers give the numbers in the quotient when written from left to right and the bracketed numbers are the decimal adjusted remainders used to find the next integer. A little practice will allow one to easily work out such divisions. Here are two more examples of divisions-

382÷37: 1→(12)→0→(120)→3→(90)→2→(160)→4 : = 10.324…

685÷81: 8→(370)→4→(460)→5→(550)→6→(640)→7 : =8.4567…

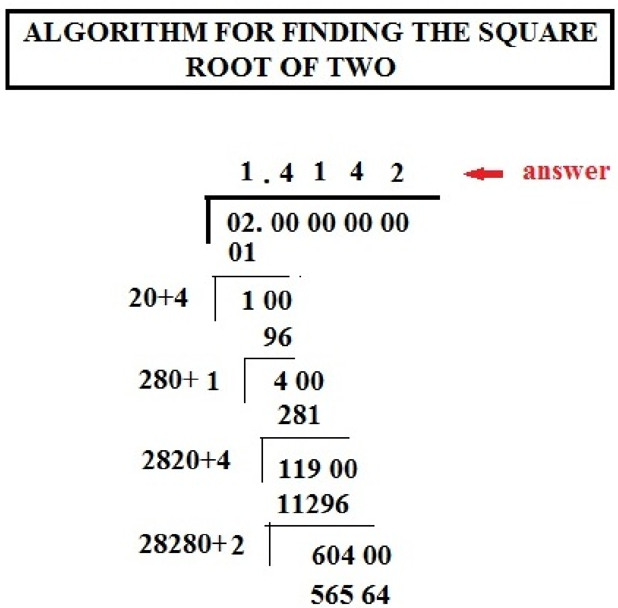
The four basic arithmetic operations discussed above may now be used in combination to work out roots, powers, and percentages of numbers. For example-

$$17^3 = 17 \cdot \{100 + 10(14) + 49\} = 17 \cdot 289 = 5780 - 3(289) = 4913$$

and the example from the introduction-

17% of \$37.78 : 20%·(37.78)-3%·(37.78)= 7.556-1.1334=\$6.42…

Obtaining the square root of a number is a bit more complicated. In pre-hand calculator days many of us were taught the following algorithm to find square roots. Neither our teacher or we students had any idea of its origin but we did know that it gave correct answers. After some seventy years I still remember how it goes. Here is the procedure for finding the root of 2-



The method is still being taught today although, for results requiring a high number of digits result, one typically uses iteration methods such as based on the Newton-Raphson method. A very simple iteration method for finding the root of two (and one which converges very rapidly and is well suited for electronic computers) is-

z[1]=1 with z[n+1]={140+99z[n]}/{99+70z[n]}

The fifth iteration already yields the 30 digit accurate result-

z[5] := 1.41421356237309504880168872420 9 ≈sqrt(2)

Let me now get back and explain why the above algorithm method for the square roots of numbers works. One begins with a three digit number N=100a+10b+c and assumes that the root of N can be written as the two digit number-

$$\sqrt{N} = 10A + B$$

Squaring this result, we have –

$$100A^2 + B(20A + B) = 100a + 10b + c$$

with the terms on the right of the equality sign being known numbers. If we now take N=625, then we get-

$$100A^2+B(20A+B) = 100\cdot 6+10\cdot 2 +1\cdot 1$$

Next choose A to be an integer lying just below sqrt(6). That is A=2. We are now left with –

$$B(40+B)=225$$

This is solved by B=5. So sqrt(625)=25. Notice how the factor 20 enters the evaluation and the required adjustment of B, just as it does in the standard division algorithm given above. For larger numbers one just repeats the present process.

SOLUTIONS TO SOME ADDITIONAL PROBLEMS

Finally let us combine many of the above procedures to calculate in our heads the resultant values.-

(1)-What is the value of the following number?

$$N = \frac{1350}{4\cdot 3^3}\sqrt{\frac{\sqrt{64\cdot 256}}{32\cdot 625}}$$

The first thing one does here is to recognize that $32=2^5$, $64=2^5$, and $256=2^7$. Also $2(1350)=100(27)-100(10^3)$ and $625=25^2$. So we have at once that-

$$N = \frac{10^2}{2^3}\left\{\frac{2}{25}\right\} = 1$$

(2)-How much will V_0 dollars be equal to in N years if it is compounded at an interest rate of i%?

Here the basic formula for compound interest is $C=V/V_0= (1+i)^N$, where V/Vo is the ratio of the starting value to the end value, i is the interest rate and N the number of years held. We have-

$$C = \frac{V}{V_0} = (1+i)(1+i)(1+i)...N\ times$$

So repeated squaring yields-

$$\begin{aligned} C &= (1+i) \quad for \quad N=1 \\ &= (1+i)^2 = 1+2i+i^2 \quad for \quad N=2 \\ &= (1+i)^3 = 1+3i+3i^2+i^3 \quad for \quad N=3 \\ &= (1+i)^4 = 1+4i+6i^2+4i^3+i^4 \quad for \quad N=4 \end{aligned}$$

So, after N years we have the series-

$$C_N = \sum_{n=0}^{N} \frac{N! i^n!}{n!(N-n)!}$$

, where 1·2·3·4· …·N is the factorial of N . If we sum things up for N=10 years and take i=0.05(5%), you get C_{10}=1.628894… . An interesting rule of thumb which follows from this result is that one will approximately double one's money when i x N≈0.7. That is you will have a 100% gain in approximately ten years when the interest rate is 7%. Unfortunately the printing of fiat money by the Federal Reserve in recent years has in effect acted as a large negative interest rate penalty on peoples savings. The zero interest rate policy and the quadrupling of the monetary base over the last six years is equivalent to having an effective negative interest rate of some 21% per year. The names of Greenspan, Bernanke, and Yellen will not be looked upon favorably in history.

(3)-You go to Las Vegas and decide to place just three bets on red on an American roulette wheel. You start with $100. What are your chances that all three bets will be in your favor and you end up with $800?

In American Roulette there are 18 red and 18 black pockets. Also there are 2 green pockets for the house. The chance of winning the first time will be 18/38=47.368421%. The chance on winning in subsequent turns are still 18/38 each time. So the chance C of coming up with red three times in a row will just be

$$C = \left\{\frac{18}{38}\right\}^3 = \left\{\frac{9x81}{19x19^2}\right\} = \frac{729}{19x361} = \frac{2.01939...}{19} = 0.10628...$$

The house's advantage will be 2/38=5.26% on each run. If you're going to gamble with roulette it is better to go to Europe (say Monte Carlo) where the wheel has only one green pocket so that the house's advantage drops to 1/37=2.702%. It will allow you to play longer before going broke. Overall, Roulette can be considered a suckers game with the cards stacked against the player. I prefer to confine my gambling to the stock market where the odds for a well versed investor are in his favor if he avoids rapid buy and sell orders which tend to make only brokers rich.

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