PROPERTIES OF NUMBER FRACTIONS

Several years ago while studying prime and composite numbers we came up with a new point function defined by-

\[ f(N) = \left[ \frac{\sigma(N) - N - 1}{N} \right] \]

where \( \sigma(N) \) is the divisor function representing the sum of all the divisors of \( N \). We refer to \( f(N) \) as the **Number Fraction**. It represents a unique fraction for any composite number and has zero value whenever \( N \) is a prime. For instance, \( f(6) = (2+3)/6 = 5/6 \) and \( f(7) = 0/7 = 0 \). It is our purpose here to discuss some additional properties of this function including when \( N \) is represented as the product of several different primes taken to specified powers.

As a starting point consider the following list-

\[ f(2) = 0 \quad , \quad f(2^2) = \frac{1}{2} \quad , \quad f(2^3) = \frac{3}{4} \quad , \quad f(2^4) = \frac{7}{8} \]

One can generalize this list at once to obtain the generic formula-

\[ f(2^n) = \left\{ 1 - \frac{1}{2^{n-1}} \right\} \]

So, we have that \( f(1024) = f(2^{10}) = \{1-(1/512)\} = 511/512 \). If we go on to \( N=3 \), one obtains the new list-

\[ f(3) = 0 \quad , \quad f(3^2) = \frac{1}{3} \quad , \quad f(3^3) = \frac{4}{9} \quad , \quad f(3^4) = \frac{13}{27} \]

Generalizing, we get-

\[ f(3^n) = \frac{1}{2} \{1 - \frac{1}{3^{n-1}} \} \]

Continuing on to powers of the next prime at \( N=5 \), we find-
These results in turn lead to the further recognitions that for all primes \( N=p \), the number fraction has the quotient value:

\[
f(p^n) = \frac{1}{(p-1)} \left\{ 1 - \frac{1}{p^{n-1}} \right\}
\]

So we see at once that for \( p=7 \) and \( n=6 \), we get the number fraction \( f(117649)=2801/16807 \). Also you will note that \( f(N^2)=1/N \) for all primes \( N=p \), but not for composites.

The next question which arises is what is the value of \( f(N) \) when \( N \) is a composite? Take first the case of \( N=6 \). Here:

\[
f(6) = f(2 \cdot 3) = \frac{(2 + 3)}{6} = \frac{5}{6}
\]

This result suggests that any semi-prime \( N=pq \) has as its number fraction:

\[
f(pq) = \frac{p+q}{pq} = \frac{1}{p} + \frac{1}{q}
\]

So \( f(77)=18/77 \). When \( N \) equals to the product of three distinct primes, the \( f(N) \) formula takes on the more complicated form:

\[
f(pqr) = \frac{(p + q + r) + pq + qr + pr}{pqr} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{pr} + \frac{1}{pq} + \frac{1}{qr}
\]

This last result says that \( f(105)=f(3\times5\times7)=(3+5+7+15+35+21)/105=86/105 \). Note that these product formulas will not work if two of the primes in the product are equal. Thus \( f(12)=f(2\times2\times3)=15/12 \) while the above formula would yield the wrong answer \( f(12)=23/12 \). The correct generic formula is:

\[
f(p^2r) = \frac{(p+r)(1+p)}{p^2r}
\]

Hence \( f(363)=f(11\times11 \times 3)=56/121 \). The simplest way to generate generic formulas of the above type is to first carry out an expansion in terms of low value primes, such as \( p=2, 3, 5 \), and then generalize the result. Doing so using \( p=2 \) and \( q=3 \), produces, upon generalization, the formula –
Applying things to \( N = 43681 = 11^2 \times 19^2 \), we find-

\[
f(43681) = \frac{(132) + (380) + 209(31)}{43681} = \frac{6991}{43681}
\]

From the above results it is now clear that if one decomposes \( N \) into product of prime numbers taken to specified powers, an evaluation of \( f(N) \) always becomes possible. The general formula for finding \( f(N) \) when \( N = p^n q^m \) can be worked out. It yields-

\[
f(p^n q^m) = \sum_{k=1}^{n} p^k + \sum_{k=1}^{m} q^k + (pq)(1 + p + p^2 + \ldots + p^{n-1} + q + q^2 + \ldots + q^{m-1}) + (pq)^2(1 + p + \ldots + p^{n-2} + q + \ldots + q^{m-2}) + O(pq)^3
\]

Here \( p \geq q \) and the expansion stops with \( (pq)^{m-1} \). Let us use this result calculate \( f(216) = f(2^3 \times 3^3) \). The formula for \( f(p^n x q^m) \) reads-

\[
f(p^n q^m) = \frac{(p + p^2 + p^3) + (q + q^2 + q^3) + pq(1 + p + p^2 + q + q^2) + (pq)^2(1 + p + q)}{(pq)^3}
\]

On setting \( p = 2 \) and \( q = 3 \) it yields-

\[
f(216) = \frac{14 + 39 + 6(19) + 36(6)}{216} = \frac{383}{216}
\]

We have noticed in earlier notes that the average value of \( f(N) \) over a specified range of \( N \), increases only very slowly with increasing \( N \). Also that when looking at graphs of \( f(N) \) over a given range of \( N \), there are certain values for which \( f(N) \) exhibits a local maximum. When this maximum exceeds about \( f(N) = 2 \), we refer to the \( N \) as a **Super-Composite**. As the name implies, super-composites tend to have a large number of divisors. A good example of a super-composite is \( N = 129600 = 2^6 \times 3^4 \times 5^2 \). It has \( f(N) = 2.675740741 \ldots \) and a total 105 distinct divisors. A graph of \( f(N) \) versus \( N \) in the neighborhood of this super-composite follows-
We see from the graph how the $f(N)$ for a super-composite towers above its neighbors. Also we note the two primes near this $N$ where the number fraction vanishes. Also the approximate symmetry in $f(N)$ about the maximum is noted.

There are an infinite number of other super-composites just as there are an infinite number of primes.

Here is a small table of super-composites and the location of the nearest prime to them:

<table>
<thead>
<tr>
<th>Super-Composite N</th>
<th>Number Fraction $f(N)$</th>
<th>Nearest Prime</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^7 \times 3^3 \times 5^5 = 3888000$</td>
<td>2.724259002...</td>
<td>N-1</td>
</tr>
<tr>
<td>$2^{13} \times 3^5 \times 5^4 \times 7 = 15676416000$</td>
<td>3.277943856...</td>
<td>N-1</td>
</tr>
<tr>
<td>$2^{17} \times 3^{11} \times 5^5 \times 7 = 20316635136000$</td>
<td>3.278832769...</td>
<td>N+1</td>
</tr>
<tr>
<td>$2^{19} \times 3^5 \times 5 = 637009920$</td>
<td>2.595058298...</td>
<td>N+1</td>
</tr>
</tbody>
</table>

We see that all of these four super-composites have $f(N)>2.5$ and possess the very interesting property that primes lie in their immediate neighborhood. Notice the power of the primes entering the definition for the Ns. The powers of the primes decrease with increasing prime. This suggests the following generalized form of the classical Mersenne Formula $2^p-1$ for generating primes:
$$[2^{p_1} \cdot 3^{p_2} \cdot 5^{p_3} \cdot \ldots \cdot ] \pm b \quad \text{where} \quad p_1 > p_2 > p_3$$

, and ‘b’ is a small positive integer. Look at \( N = 2^{13} \cdot 3^7 \cdot 5^3 = 2239488000 \). This is a super-composite with \( f(N) = 2.743200874 \ldots \) and one finds a prime at \( N-7 = 2239487993 \).

Semi-primes \( N=pq \) typically will have values of \( f(N) \) just slightly above zero and never will approach super-composite values. Take the primes \( p = 6121 \) and \( q = 22003 \). These produce the semi-prime \( N = 134680363 \) for which \( f(N) = 0.002088203460 \ldots \). Such low values of \( f(N) \) help in identifying semi-primes.

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