FURTHER PROPERTIES OF THE NUMBER FRACTION FUNCTION

In several recent notes we have introduced a new function $f(N)$ which we have termed the Number Fraction. It is related to the classic sigma function of Number Theory but has the advantage over the former in that it appears to be bounded for all positive integers and vanishes whenever the number is a prime. Its definition is:

$$f(N) = \frac{\sigma(N) - (N + 1)}{N}$$

Recalling that the sigma function just represents the sum of all divisors of a number $N$, we see that the Number Fraction function eliminates the first and last of the divisors and then divides the result by $N$. Thus the number $N=24$ has a Number Fraction $f(24) = (2+4+6+8+12)/24 = 4/3$. If one plots the Number Fraction versus $N$ one gets the following graph:

![Number Fraction Graph](image)

The interesting thing one notices from the graph is that $f(N)$ will have values greater than 1 mainly when $N$ is a multiple of 6. We call such numbers super-composites as they have many divisors. One also sees that the function vanishes when $N$ is a prime. We call these primes the Q Primes. They represent essentially all primes above $p=3$ and are shown in blue at the bottom of the graph. They can all be represented by $6n\pm1$. Semi-Primes $N=pq$ typically are found to have very small but not zero value for $f(N)$. Thus the semi-prime $N=260497=331x787$ has the non-zero value of $f(N)=1118/260497=0.00429…$ It is our purpose here to derive some additional properties of the Number Fraction $f(N)$. 
As a starting point we look at $f(2^n)$ for $n=1, 2, 3, \text{ through } 10$. This yields the table:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$2^n$</th>
<th>$f(2^n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1/2</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>3/4</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>7/8</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>15/16</td>
</tr>
<tr>
<td>6</td>
<td>64</td>
<td>31/32</td>
</tr>
<tr>
<td>7</td>
<td>128</td>
<td>63/64</td>
</tr>
<tr>
<td>8</td>
<td>256</td>
<td>127/128</td>
</tr>
<tr>
<td>9</td>
<td>512</td>
<td>255/256</td>
</tr>
<tr>
<td>10</td>
<td>1024</td>
<td>511/512</td>
</tr>
</tbody>
</table>

We see at once that this implies -

$$f(2^{n+1}) = \frac{(2^n - 1)}{2^n} \text{ or the equivalent } f(2^{n+1}) = \frac{(f(2^n) + 1)}{2}$$

It shows that we are dealing with a bounded function which approaches unity as $n$ approaches infinity. If one makes a similar expansion for $3^n$ it is found that –

$$f(3^{n+1}) = \left(\frac{1}{3}\right)[f(3^n) + 1]$$

Next trying this for $N=4^n$, shows that $f(4^{n+1})\neq(1/4)[f(4^n)+1]$. From this we can infer the important new identity-

$$f(p^{n+1}) = \frac{f(p^n) + 1}{p}$$

where $p=2, 3, 5, 7, 11, 13, \ldots$ are the prime numbers. This equality will not work for composites!. To verify that it works for any prime, consider $p=5$ and $n=3$. In this case we have $f(5^3)=31/125$ and $f(5^3)=6/25$, so $(6/25+1)/5=31/125$ which checks. Trying this for a composite number such as $N=12$ won’t work. For there $f(12^3)\neq[f(12^2)+1]/12$. The identity allows us to introduce the new **prime number test** –

**A number $p$ is prime if $p=[f(p^n)+1]/f(p^{n+1})$, otherwise it is a composite**

Lets run this test for the Fermat Number $N=2^{32}+1=4294967297$. Taking $n=2$, we have –
This is not equal to $2^{32} + 1$ so $N$ is a composite. Leonard Euler first showed this number to be a composite after spending months to come up with an answer. It is coincidence that the right side in this calculation yields a value close to the factor 641 of $N$. We have not seen this happen again for other semi-primes such as 77 or 493.

Another property of the $f(N)$ function follows from the fact that any positive integer can be expressed as the product of primes taken to specified powers $'a'$. One knows from Number Theory that –

$$
\sigma(N) = \prod_{r=1}^{m} \left( \frac{p_r^{a_r+1} - 1}{p_r - 1} \right), \text{ where } p_r \text{ are the prime components of } N \text{ and } a_r \text{ their powers}
$$

The term inside the product sign is just the sum of the finite geometric series-

$$
\sum_{j=0}^{a_r} p_r^j = 1 + p_r + p_r^2 + p_r^3 + \ldots + p_r^{a_r}
$$

Using the above product form for $\sigma(N)$, we find-

$$
f(N) = \prod_{r=1}^{m} \left( \frac{p_r - p_r^{-a_r}}{p_r - 1} \right) = \frac{(N+1)}{N}
$$

since we also have that-

$$
N = \prod_{r=1}^{m} p_r^{a_r}
$$

Thus, if we choose $N=882=2 \cdot 3^2 \cdot 7^2$, we find that $f(882)$ equals-

$$
f(882) = \left( \frac{2 - 2^{-1}}{2 - 1} \right) \left( \frac{3 - 3^{-2}}{3 - 1} \right) \left( \frac{7 - 7^{-2}}{7 - 1} \right) \frac{883}{882} = \frac{670}{441}
$$

Also we see that for any prime taken to the nth power, we have-

$$
f(p^n) = \frac{(p^n - 1)}{(p - 1)p^{n-1}} = \frac{1 - p^{1-n}}{p - 1}
$$
So if we set p=7 and n=4, we find f(2401)=57/343. The reduction to the final quotient form shown is easiest to achieve by taking the gcd(342,2058)=6. An additional result valid for all semi-primes \( N=pq \) is -

\[
f(pq) = \frac{p+q}{pq}
\]

Taking \( p=65353 \) and \( q=125131 \) says \( f(8177686243)=0.000023293141… \) An obvious extension is that-

\[
f(p_1 \cdot p_2 \cdot p_3 \cdots p_m) = \left( \frac{1}{N} \right) \prod_{k=1}^{m} (p_k + 1) - 1
\]

where \( N=p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdots p_m \) and the \( p_k \)'s are all primes. Thus if \( N=2(3)(5)(7)(11)(13)=30030 \), we find-

\[
f(N) = \frac{1}{N} \left\{ 3 \cdot 4 \cdot 6 \cdot 8 \cdot 12 \cdot 14 - 1 \right\} = \frac{6067}{2730}
\]

One notices from the graph given earlier that the largest values of \( f(N) \) seem to occur when \( N \) is a multiple of 6. Under these conditions one encounters the largest super-composites. Let us see if we can develop a general formula for this case where \( N=6n \). We first construct another table, writing down the values of \( f(6n) \) for \( n=1 \) through 10-

<table>
<thead>
<tr>
<th>6n</th>
<th>f(6n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>5/6</td>
</tr>
<tr>
<td>12</td>
<td>5/4</td>
</tr>
<tr>
<td>18</td>
<td>10/9</td>
</tr>
<tr>
<td>24</td>
<td>35/24</td>
</tr>
<tr>
<td>30</td>
<td>41/30</td>
</tr>
<tr>
<td>36</td>
<td>3/2</td>
</tr>
<tr>
<td>42</td>
<td>53/42</td>
</tr>
<tr>
<td>48</td>
<td>25/16</td>
</tr>
<tr>
<td>54</td>
<td>65/54</td>
</tr>
<tr>
<td>60</td>
<td>107/60</td>
</tr>
</tbody>
</table>

There is no noticeable pattern in these values of \( f(6n) \) other than that all but the first have \( f(6n)>1 \) and hence are super-composites. Especially large values are found for 6n= 36, 48, and 60. These are all divisible by 12. Continuing on we find \( f(72)=61/36, f(84)=139/84 \), and \( f(96)=155/96 \). Still no pattern emerges, meaning that \( f(12n) \) must be recalculated for every \( n \). The only thing one notices is that \( f(6n) \) is probably bounded by a small number, no matter what the value of \( n \) is. For example-
Another property of the $f(N)$ function is that we can express its difference as follows:

$$f(N) - f(M) = \frac{\sigma(N)}{N} - \frac{\sigma(M)}{M} + \frac{(N-M)}{NM}$$

The sigma function (being built into most advanced mathematics computer programs) allows us to quickly calculate the difference between $f(N+1)$ and $f(N)$, namely –

$$f(N+1) - f(N) = \frac{\sigma(N+1)}{N+1} - \frac{\sigma(N)}{N} + \frac{1}{N(N+1)}$$

Since $\sigma(30) = 72$ and $\sigma(29) = 30$, we have that $f(30) - f(29) = f(30) = 41/30$. Remember that $N=29$ is a prime number.

We next look at $f(60) - f(48)$. From the last table we have -

$$f(60) - f(48) = \frac{107}{60} - \frac{25}{16} = \frac{53}{240}$$

In the above formula we use $\sigma(60) = 168$ and $\sigma(48) = 124$ to get -

$$\frac{168}{60} - \frac{124}{48} + \frac{12}{48(60)} = \frac{53}{240}$$

which agrees.

We can also try to sum successive $f(N)$ functions. This produces -

$$\begin{align*}
f(1) &= 0 \\
f(1) + f(2) &= 0 \\
f(1) + f(2) + f(3) &= 0 \\
f(1) + f(2) + f(3) + f(4) &= 1/2 \\
f(1) + f(2) + f(3) + f(4) + f(5) &= 1/2 \\
f(1) + f(2) + f(3) + f(4) + f(5) + f(6) &= 4/3 \\
f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7) &= 4/3 \\
f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7) + f(8) &= 25/12 \\
f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7) + f(8) + f(9) &= 29/12 \\
f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7) + f(8) + f(9) + f(10) &= 187/60
\end{align*}$$

There is no obvious pattern to these sums other than that they increase with increasing $n$. Calling the sum of the first $n$ terms $S(n)$ we have that $S(n+1)$ is given by the iterative formula –
\[ S(n+1) = S(n) + f(n+1) \quad \text{subject to } S(3) = 0 \]

We can quickly evaluate this formula using the one line program-

\[
\text{x}[3] := 0; \text{ for } n \text{ from 3 to 400 do } x[n+1] := \text{evalf}(x[n] + (\sigma(n+1) - (n+2))/(n+1)) \text{ od;} 
\]

It produces the sums \( S[50] = 26.556247 \ldots, S[100] = 58.035255 \ldots, S[200] = 121.825035 \ldots \) and \( S[400] = 249.547183 \ldots \). So the sums grow very slowly and approximately doubles in size as the value of \( n \) is doubled. The sum \( S[N] \) undoubtedly diverges as \( N \) approaches infinity but the individual values of \( f(N) \) probably remains finite. So far the largest value we have been able to find is \( f(15135120) = 3.707292641 \ldots \). Numbers which seem to have the largest values for \( f(N) \) for larger \( N \) are found to be integer multiples of \( 840 = 6(140) = 2^3 \cdot 3 \cdot 5 \cdot 7 \). The numbers within one unit of such a large \( f(N) \) are often primes. This will be the case for the prime \( p = 15135121 \).

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