MORE ON APPOXIMATIONS FOR TANGENT AND OTHER TRIGONOMETRIC FUNCTIONS

About a decade ago we were playing around with integrals involving even Legendre Polynomials P(2n,x). One of these integrals led to the following identity-

$$I=\int_{x=0}^{1}\frac{P_{2n}(x)}{1+x^2}dx=-M(n,1)+N(n,1)(\frac{\pi}{4})$$

Here-

$$M(n,1)=\mathrm{int}(quo(P(2*n,x),x^2+1,x),x=0..1);$$

and-

$$N(n,1)=rem(P(2*n,x),1+x^2,x);$$

The first of these represent the integral of the quotient function without remainder and the second the remainder term when both are expressed in MAPLE language. For n=4 this produces the identity-

$$0.0001994=(1667/32)\pi-(5728/35)$$

We note that the left hand side of this expression is very small and indeed will approach zero as n gets large. So we get the approximation that-

$$\pi=4M((n,1)/N(n,1)$$

as n>>1. For the above case of n=4 we get the inequality-

$$\pi>3.1415888$$

compared to the exact value of π=3.14159265… The larger n becomes the closer the π quotient approaches the exact value.

The above result led us to the conclusion that one can use various other denominator functions not just $1+x^2$ to get approximations for an infinite number of other functions. One such important set are the trigonometric functions tan(a), sin(a) and cos(a). Since we have the identities –

$$\sin(a) = \frac{1}{\csc(a)} = \frac{\tan(a)}{\sqrt{1+\tan(a)^2}} \quad and \quad \cos(a) = \frac{1}{\sec(a)} = \frac{1}{\sqrt{1+\tan(a)^2}}$$

, we need only develop the approximations involving tan(a).

To do this we can use the following two integrals as a starting point-

$$I = \int_{x=0}^{1} P(2n,x)\cos(ax)dx \quad or \quad J = \int_{x=0}^{1} P(2n+1,x)\sin(ax)dx$$

Note that the form of the Legendre polynomials in the product term in the integrands was so chosen as to keep the product an even function. If this is not done then the expansion will produce a third function in addition to N and M.

Carrying out the appropriate N and M evaluations for sin(a) and cos(a) produces the following first eight approximations for tan(a) -

$$T(1,a) = a$$

$$T(2,a) = \frac{3a}{3-a^2}$$

$$T(3,a) = \frac{15a-a^3}{15-6a^2}$$

$$T(4,a) = \frac{105a-10a^3}{105-45a^2+a^4}$$

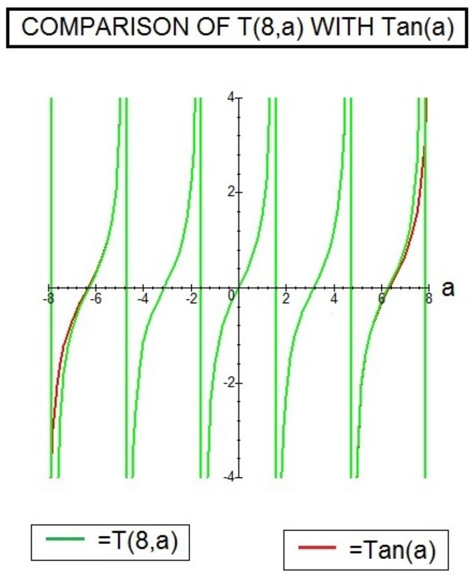
$$T(5,a) = \frac{945a-105a^3+a^5}{945-420a^2+15a^4}$$

$$T(6,a) = \frac{10395a-1260a^3+21a^5}{10395-4725a^2+210a^4-a^6}$$

$$T(7,a) = \frac{135135a-17325a^3+378a^5-a^7}{135135-62370a^2+3150a^4-28a^6}$$

$$T(8,a) = \frac{2027025a-270270a^3+6930a^5-36a^7}{2027025-945945a^2+51975a^4-630a^6+a^8}$$

The larger n becomes in T(n,a) the more accurate the approximation will be. In all cases we note that tan(a)/a approaches unity as 'a' gets very small. Also, by having the denominator vanish in these expressions, estimates for the infinities of tan(a) can be found. For example, setting the denominator in T(4,a) to zero produces a=1.57123.. which lies close to the true pole at π/2=1.57079.. Plotting the approximation T(8,a) over the range -8<a<8yields the following picture-



The agreement is excellent for |a|<6, including showing the infinity at ±π/2 and ±3π/2 and zero at 0,±π and ±2π.

To construct an accurate table for the trigonometric functions it is necessary to only know the value of tan(a) in the limited range 0<a<π/4 since we have the well known identities-

$$\tan(\frac{\pi}{4}-\theta)\tan(\frac{\pi}{4}+\theta)=1 \quad \mathit{and} \quad \tan(\theta)=-\tan(-\theta)=\tan(\pi+\theta)$$

which allow us to find tan(θ) everywhere outside the range 0<a<π/4 once the values inside are known.

Here is a table we have constructed using five degree intervals between a=0 and π/4. The T(8,a) approximation has been used-

a in deg	a in rads	T(8,a)≈tan(a)
0	0	0
5	π/36	0.0874886635259240052220186694349
10	π/18	0.176326980708464973471090038
15	π/12	0.267949192431122706472553
20	π/9	0.363970234266202361351
25	5π/36	0.466307658154998559
30	π/6	0.577350269189625764
35	7π/36	0.700207538209709
40	2π/9	0.839099631177
45	π/4	0.9999999999999

These T(8,a) approximations have their digits given out to the point where the values depart from the exact value for tan(a). Note that the table is accurate to at least 12 digits. The smaller 'a' or larger n gets the more accurate the T(n,a) approximation becomes. One can improve the accuray for tan(a) by going to larger values of n in T(n,a) or better look at departures from a known point nearby and then evaluating T(8,a) of the difference. We demonstrate such an approach for finding tan(32deg) using the known value of tan(π/6)=1/sqrt(3) and then evaluating T(8,π/90). Using the tangent formula for the sum of two angles we have approximately-

$$\tan(32\deg) = \frac{\frac{1}{\sqrt{3}} + T(8,\pi/90).}{1 - \frac{1}{\sqrt{3}} T(8,\pi/90)} = 0.62486935190932750978051082794943665 83$$

This result is accurate to 37 places.

To get corresponding value of sin(a), cos(a), sec(a), and csc(a) we need only use their form expressed in terms of the tangent. So for a=22.5deg=π/8rads we get-

$$\sin(\pi/8) \approx \frac{T(8,\pi/8)}{\sqrt{1+T(8,\pi/8)^2}} = 0.38268343236508977172$$

$$\cos(\pi/8) \approx \frac{1}{\sqrt{1+\mathrm{T}(8,\pi/8)^2}} = 0.92387953251128675613$$

$$\csc(\pi/8) = \frac{\sqrt{1+T(8,\pi/8)^2}}{T(8,\pi/8)} = 2.6131259297527530557$$

$$\sec(\pi/8) = \sqrt{1+T(8,\pi/8)^2} = 1.0823922002923939687$$

These results are accurate to at least 19 digits when compared to the exact solution. Note that here we can express the cosine in closed form as-

cos(π/8)=sqrt[1+sqrt(2)]/sqrt(2*sqrt(2))

To get even higher accuracy all that is necessary to do is increase the 8 in T(8,a) to say 12.

Finally we say a few words of why the present technique gives such excellent approximations. It stems from the fact that Legendre Polynomials P(n,x) have n/2 zeros in 0<x<1 and so are rapidly oscillating functions. Multiplying a slower varying function into P(n,x) causes a great deal of cancelations thus making the product integral progressively smaller as n is increased. Other periodic functions such as the Chebyshev Polynomials could also be used, but these have been found to be less effective when integrating over the range 0<x<1.

U.H.Kurzweg
Nov.15, 2018
Gainesville, Florida