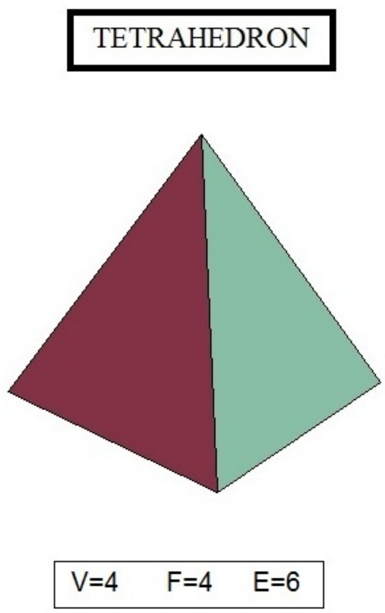
MORE ON THE FIVE PLATONIC POLYHEDRA?

The ancient Greeks were already familiar with the five solid figures known as the Platonic Polyhedra. The are defined as solid figures bounded by identical faces in the form of equilateral triangles , squares or regular pentagons. They start with the tetrahedron whose four bounding surfaces are identical equilateral triangles, the hexahedron bounded by six identical square surfaces, the octahedron with eight bounding surfaces of equal sized equilateral triangles, the dodecahedron bounded by twelve pentagonal surfaces, and the icosahedron bounded by twenty equilateral triangles. It is our purpose here to discuss the properties of these five Platonic Solids in further detail.

TETRAHEDRON:

This solid represents the simplest of the Platonic figures. It consists of just four equilateral triangle faces (F=4) with six edges (E=6) edges and four vertices (V=4). Here is a graph of a tetrahedron-



The polyhedron satisfies the famous Euler Formula -

$$V+F-E=2$$

which works for any convex polyhedron. The area of a single equilateral triangle equals (1/2)sh, where s is the length of a side, and h=sqrt(3)s/2 This yields a total tetrahedron surface area of-

$$\text{Area}=\text{sqrt}(3)s^2=1.732050\ s^2$$

The volume equals four times the product of the (base area)/3 times the height H of the pyramid measured upward from the centroid of the base. This yields-

$$Volume = (\frac{\sqrt{3}}{4}s^2)(\frac{1}{3})(\sqrt{\frac{2}{3}}s) = \frac{\sqrt{2}}{12}s^3 = 0.1178511\,s^3$$

The surface to volume ratio is here-

$$Ratio = \frac{Surface\ Area}{Volume} = \frac{12\sqrt{3}}{\sqrt{2}s} = \frac{14.696938}{s}$$

We should also mention the tetrahedral numbers defined as 1-4-10-20-35- and given in generalized form as-

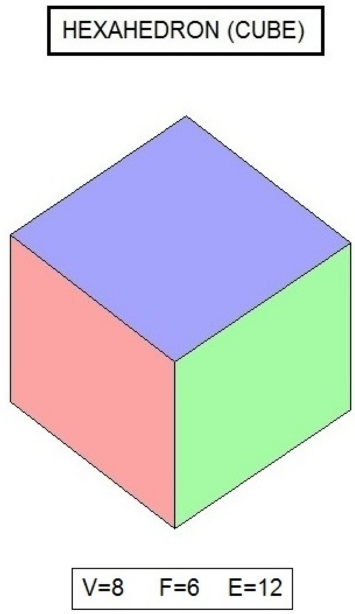
$$T(n) = \frac{n(n+1)(n+2)(n+3)}{6}$$

represent the total number of spheres which can be stably stacked above each other. That is, a triangular base of three spheres can hold one sphere in the second layer for a total of four spheres. If we start with a base of six spheres placed in a triangular pattern this will allow placing three spheres in the second layer and one in the third layer for a total of 10 spheres. If one continuous on with this type of layer stacking the total number of spheres up through the nth layer will be T(n). As T(n) gets large the resultant stacked spheres configuration will resemble essentially a regular tetrahedron.

Certain crystal structures also have tetrahedral shape. The requirement for their appearance is that four equal sized atoms are placed at the four vertices of a regular tetrahedron.

HEXAHEDRON(CUBE):

The hexahedron also known as the cube is the second of the five Platonic solids. Its properties are the easiest to understand. It has F=6, V=8, and E=12. This again agrees with the Euler Formula that 8+6-12=2. The total surface area is just Area=$6s^2$, where s is the edge length. The volume is Volume=s^3. The surface to volume ratio equals 6/s. Here is a graph of this solid-



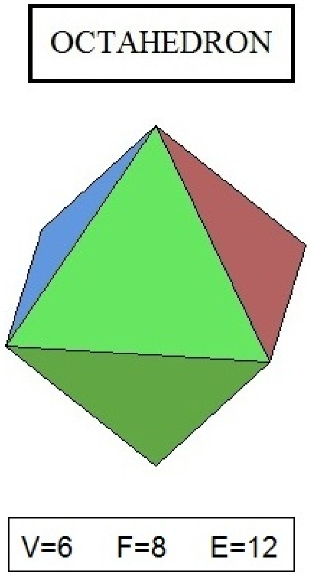
An alternative way to calculate the cube's volume (not necessary here but of use for the upcoming additional Platonic solids), is to look at this polygon as the sum of six pyramids whose vertices lie at the cube center. Since we know that any pyramid has a volume equal to (base · height)/3 and that for a cube H=(1/2)s, we get the total cube has a volume six times this, namely,-

$$Vol=6(s^2)(s/2)/3=s^3$$

as before.

OCTAHEDERON:

This Platonic Polyhedron has a total of eight equilateral triangle faces(F=8) with twelve edges(E=12) and six vertices(V=6). It satisfies the Euler Equation V+F-E=6+8-12=2. its graph looks like this-



Its surface area is 8[sqrt(3)/4]. That is-

$$Area = 2\sqrt{3}\, s^2 = 3.46410\, s^2$$

The volume is easiest to determine by looking at this solid as two identical square base pyramids of height h=2/sqrt(2) each. The resultant volume is-

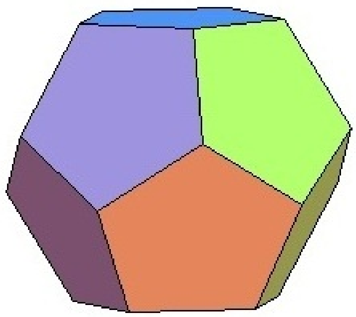
$$Volume = \frac{\sqrt{2}s^3}{3} = 0.47140\, s^2$$

The area to volume ratio is R=3sqrt(6)/s.

DODECAHEDRON:

This is perhaps the more interesting Platonic solid. It is bounded by twelve pentagonal faces(F=12) with twenty vertices(V=20) and thirty edges(E=30). One has 12+20-30=2 . A graph of a dodecahedron follows-

DODECAHEDRON



V=20 F=12 E=30

To calculate its total surface area one simply needs to multiply the area of a pentagon by twelve. Since the area of a regular pentagon is $dA=5s^2\cot(\pi/5)/4$, we get-

$$Area = 12(\frac{5s^2}{4})\cot(\frac{\pi}{5}) = 20.64573\, s^2$$

To get the volume of the dodecahedron we add together the volume of twelve pyramids of pentagonal base and height h of the distance between the center of a polygon surface to the dodecahedron centroid. After some trignometric manipulations one finds that-

h=(s/2)sqrt[5/2+11sqrt(5)/10]=1.113516365 s

Thus the dodecahederon has the total volume of-

Volume=12 dA h/3 s^3=7.663116455 s^3

Here h also represents the radius of a sphere inscribed within the dodecahedron.
The surface area-volume ratio becomes R=2.69416/s and so approaches close to this ratio for a sphere where R=3/s.

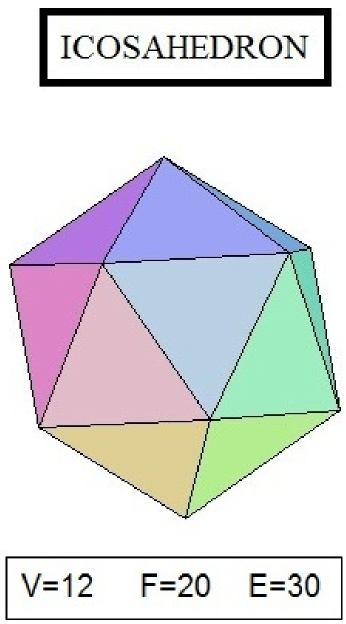
Several years ago we constructed a dodecahedron from twelve equal sized pentagonal plates in our wood work shop. Here is the resultant model using plywood with walnut veneer-

Recently there have appeared several stories on the internet concerning the hundreds of metal dodecahedra with different sized holes found at ancient Roman archeological sites in Great Britain and Germany and other parts of Western Europe.
Here are two pictures of these baseball sized devices-

No one has been able to figure out the purpose for these structures although there have been broad speculations that they were used for range finders, candle holders, or astronomical measuring instruments. The different sized holes in these bronze figures also defy explanation. My own view is that they were religious devices used to aide meditation and for possible health cures. A small burning candle inside one of these polyhedra would make for a mysterious aura. The spiritual aspect of these devices is further supported by the fact that many of these objects were buried with their owners in this pre-Christian era. Plato already mentions well over two thousand years ago that he thought the dodecahedron was used by god to arrange the constellations in the heavens. A second possibility is that these devices were used to mark the twelve months of the new Julian calendar. The small metal protrusions could serve to stabilize and orient the dodecahedron for any given lunar month.

ICOSAHEDRON:

This is the last of the Platonic solids. It has 12 vertices, 20 equilateral triangle faces and a total of 30 edges. It looks as follows-



Each triangle face has an area of A=[sqrt(3)/4]s^2 leading to a total surface area of-

$$\text{Area}=5\text{sqrt}(3)\ s^2=8.660254\ s^2$$

To get the volume we treat one of the triangles as the base of a pyramid of heigt h. Here a little geometry shows that h=5s(3+sqrt(5)/[12sqrt(3)]=1.25960s. So the total volume of the icosahedron becomes-

$$\text{Volume}=(5/12)[3+\text{sqrt}(5)]s^3=2.18170\ s^3$$

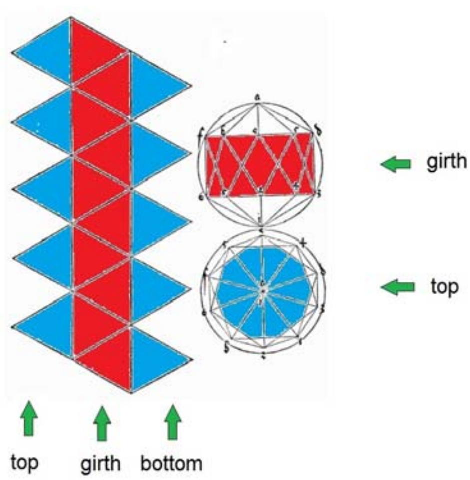
The ratio of surface area is R=3.969498/s.

Here is a wooden model of an icosahedrons I constructed about a decade ago. It took twenty equal sized equilateral plywood triangles covered with cherry veneer to build the structure.

WOOD CONSTRUCTION OF AN ICOSAHEDRON

It is also possible to construct a icosahedron from cardboard by folding the following pattern of triangles-

ALBRECHT DURER'S 1525 NET FOR CONSTRUCTING AN ICOSAHEDRON



This construction was first proposed by the famous German artist Albrecht Durer in his 1525 book on perspective. The Japanese are quite familiar with constructing all sorts of three dimensional structures by folding 2D surfaces(Origami).

U.H.Kurzweg
September 2, 2018
Gainesville, Florida