MORE ON THE PROPERTIES OF THE SINE INTEGRAL

In an earlier article (Oct. 2011) we discussed some of the properties of the sine integral defined as-

$$ Si(x) = \int_0^x \frac{\sin(x)}{x} \, dx $$

We want here to revisit this function and develop some of its more advanced features. First of all using complex variables we can write Si(x) as-

$$ Si(x) = \text{Im} \int_0^x \frac{\exp(ix)}{x} \, dx = \frac{\pi}{2} - \text{Im} \int_x^\infty \frac{\exp(ix)}{x} \, dx $$

The \( \pi/2 \) in this expression comes from the fact that \( Si(\infty)=\pi/2 \). On integrating by parts we also have-

$$ Si(x) = \frac{\pi}{2} - \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} + \int_x^\infty \frac{\sin(t)}{t^3} \, dt $$

So for large \( x \) the function \( Si(x) \) is approximated by the above expression without the integral. It oscillates between positive and negative values relative to \( \pi/2 \) as \( x \) gets large.

The sine integral value for small \( x \) is best seen via a simple infinite series expansion-

$$ Si(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \frac{x^3}{3(3!)} + \frac{x^5}{5(5!)} - \frac{x^7}{7(7!)} + \ldots $$

It shows that \( Si(x) \approx x-x^3/18 \) for \( x<<1 \).

To bridge the gap between the \( x<<1 \) approximation and the \( x>>1 \) approximation, we need to go to an integral form for \( Si(x) \). We can write, as earlier, that-

$$ Si(x) = \frac{\pi}{2} - \text{Im} \int_{t=x}^\infty \frac{\exp(it)}{t} \, dt $$
and then make the substitution $t=\exp(\mathrm{i}v)$. This produces $\mathrm{d}t=\mathrm{i}\exp(\mathrm{i}v)\mathrm{d}v$. From it follows the new integral form:

$$Si(x) = \frac{\pi}{2} - \int_{v=0}^{\pi/2} \exp[-x\sin(v)]\cos[x\cos(v)]\mathrm{d}v$$

Evaluating the right hand side of this equation produces the following exact values for $Si(x)$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$Si(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.4931074180</td>
</tr>
<tr>
<td>1.0</td>
<td>0.9460830704</td>
</tr>
<tr>
<td>2.0</td>
<td>1.6054129768</td>
</tr>
</tbody>
</table>

These values merge well into the asymptotic forms given above. Furthermore, the values of $Si(x)$ predicted by this formula hold for all $0<x<\infty$. A plot of $Si(x)$ together with its limiting forms follow-

![Plot of Si(x) and its approximations](image)

To get the derivatives of $Si(x)$ it is simplest to go back to the integral form of $Si(x)$ as given in the original definition. It yields at once that:
\[
\frac{d\text{Si}(x)}{dx} = \frac{\sin(x)}{x} \quad \frac{d^2\text{Si}(x)}{dx^2} = \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}
\]
\[
\frac{d^3\text{Si}(x)}{dx^3} = -\frac{\sin(x)}{x} - \frac{2\cos(x)}{x^2} + \frac{2\sin(x)}{x^3}
\]
\[
\frac{d^4\text{Si}(x)}{dx^4} = -\frac{\cos(x)}{x} + \frac{3\sin(x)}{x^2} + \frac{6\cos(x)}{x^3} - \frac{6\sin(x)}{x^4}
\]

Using the \(\cos(x)\) term in \(\text{Si}(x)''\) and plugging it into \(\text{Si}(x)'''\) produces the third order linear differential equation:

\[
x \frac{d^3y}{dx^3} + 2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 0
\]

This equation has as a solution \(y = \text{Si}(x)\) with \(y' = \sin(x)/x\). On letting \(u = y(x)'\), we also get the second order equation:

\[
xu'' + (2/x)u' + u = 0
\]

This last equation is recognized as a Bessel equation of order \(\frac{1}{2}\). One of its two solutions is –

\[
u = J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin(x) = \sqrt{\frac{2x}{\pi}} \text{Si}(x)'
\]

We also have-

\[
\text{Si}(x) = \sqrt{\frac{\pi}{2}} \sum_{r=0}^{\infty} \frac{J_{1/2}(x)}{\sqrt{x}} dx
\]

A plot of the Bessel function of the first kind of order one-half follows-
Certain integrals involving Si(x) can be solved in closed form. Take for instance-

\[
\int_{0}^{\infty} Si(t) \exp(-at) dt = \text{Laplace}[Si(t)] \quad \text{with} \quad s = a \quad \Rightarrow \quad \frac{\text{arccot}(a)}{a}
\]

\[
\int_{0}^{x} (a + bt)Si(t) dt = \left\{ ax + \frac{b}{2}x^2 \right\} Si(x) + \left\{ a + \frac{b}{2} \right\} x \cos(x) - \frac{b}{2} \sin(x) - a
\]

and

\[
\int_{0}^{1} t^2 Si(t) dt = \frac{1}{3} \left\{ Si(1) - \cos(1) - 2 \sin(1) + 2 \right\}
\]

A function related to Si(x) is the shifted form-

\[ si(x) = Si(x) - \frac{\pi}{2} \]

It has the appearance shown-
Unlike \( Si(x) \), its integral is finite when taken over \( 0 < t < \infty \). Indeed we have:

\[
\int_0^\infty si(t) dt = -1
\]

Also

\[
\int_0^\infty si(t)^2 dt = \frac{\pi}{2} \quad \text{and} \quad \int_0^\infty si(t) \exp(-at) dt = \frac{1}{a} \{ \arccot(a) - \frac{\pi}{2} \}
\]

Finally we point out that the earlier integral form for \( Si(x) \) has the integral extend over the limited range \( 0 < v < \pi/2 \). It suggest if we use a low order approximation \( T(v) \) for the tangent over this range, we can get a reasonable approximation for \( Si(x) \) for all \( x \). The approximation reads:

\[
Si(x) \approx \frac{\pi}{2} - \pi^2 \int_0^{\pi/2} \exp[-xT(v)/\sqrt{1+T(v)^2}] \cos[x/\sqrt{1+T(v)^2}] dv
\]

where \( T(v) \) ia approximated one of our earlier derived forms:

\[
T(v) = \frac{v(15 - v^2)}{(15 - 6v^2)}
\]
Plotting this approximation for $Si(x)$ over the range $0 < x < 20$ produces the following graph-

The curve, when compared with the exact $Si(x)$, is essentially indistinguishable. We can also differentiate $Si(x)$ once retaining the tan approximation $T(v)$. This yields the approximate result -

$$T(v) = \frac{v(15v^2)}{(10-6v^2)}$$

That the right hand side of this last expression is indeed a valid approximation for $\frac{\sin(x)}{x}$ can be seen in the following plot over the range $0 < x < 20$-
Here is a picture of Irma just a few hours before arriving in Gainesville. The photo looks a lot scarier than it turned out to be for us here in north-central Florida—