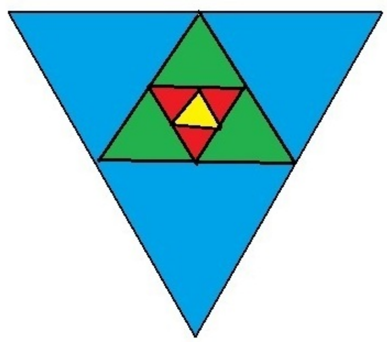
NESTING OF REGULAR POLYGONS

In mathematics nesting is referred to as placing ever smaller identical shapes into each other. Perhaps the best known example of nesting are the well known Russian wooden babushka dolls familiar to most. We want in this article to look at a special kind of nesting in which identical regular polygons are placed inside each other with the stipulation that the vertices of the next generation just touch the midpoints of the previous identical shape. The simplest example of such a nesting occurs for equilateral triangles placed into each other. The resultant figure looks as follows-

NESTING IN AN EQUILATERAL TRIANGLE



Area Ratio R between the nth and (n+1)th generation is 4

One sees from simple geometry that each generation gets smaller in area by a factor of four. So we have the Area Ratio-

$$R = \frac{area\ of\ \ Nth\ generation}{area\ of\ \ (N+1)th\ generation} = 4$$

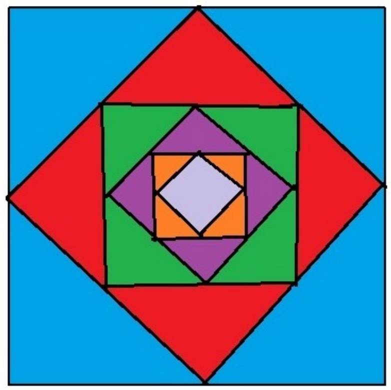
The total area generated by all generations is, via the geometric series, equal to-

$$A_{total} = A_o \sum_{n=0}^{\infty} (\frac{1}{4})^n = (\frac{4}{3}) A_{.0}$$

, where A_o is the area of the starting equilateral triangle.

A second simple example of polygon nesting is that of a square. There we have the following picture-

NESTING IN A SQUARE



Area ratio R for nth to (n+1)th generation equals 2

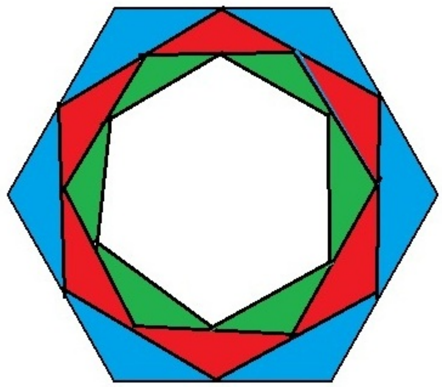
This time we see , again by simple geometry, that the area ratio between generations equals-

$$R = \frac{area\ Nth\ generation}{area\,(N+1)th\ generation} = 2$$

The Total Area of all the nested squares becomes $2A_o$.

The next simplest nesting occurs for the regular hexagon. We have there the picture-

NESTING OF A REGULAR HEXAGON



Area Ratio=4/3

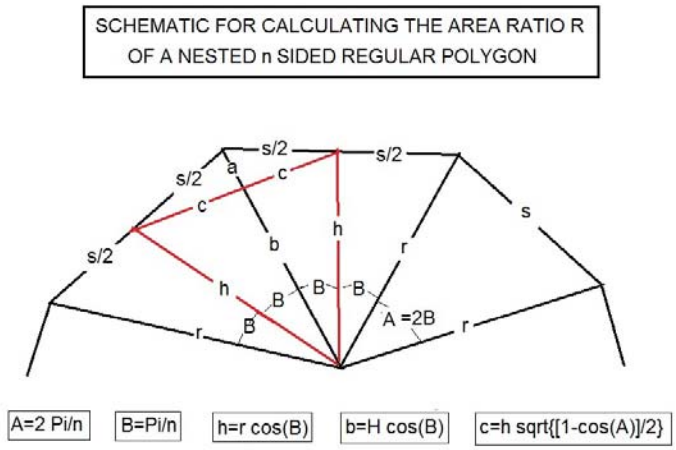
Here the area ratio is a bit more difficult to find and requires use of the law for oblique triangles. We find-

$$R = \frac{area\ Nth\ generation}{area\,(N+1)th\ generation} = 4/3$$

The total area obtained by adding up all the hexagons yields-

$$A_{total} = A_o \sum_{n=0}^{\infty} (\frac{3}{4})^n = 4A_o$$

Having determined the area ratio between generations for polygons with n=3, 4, and 6 sides, we are now in a positions to find the general area ratio for any regular polygon of n sides. Here is how this works. We start with the following schematic-



Now the area of one of the large black bordered isosceles triangles is-

$$Area_{black} = sh/2 = (\frac{s}{2})r\cos(B) = (\frac{s^2}{2})\frac{\cos(B)}{\sqrt{2[-\cos(2B)]}}$$

The area of the red bordered triangle equals-

$$Area_{red} = cb = hb\sqrt{\frac{1-\cos(2B)}{2}} = br\cos B)\sqrt{\frac{1-\cos(2B)}{2}}$$

But r=s/sqrt{2[1-cos(2B)]} and b=r[cos(B)]^2. Combining we get-

$$Area_{red} = \frac{s^2[\cos(B)]^3}{2\sqrt{2[1-\cos(2B)]}}$$

Taking the ratio we find the extremely simple result that-

$$R = \frac{Area\ Nth\ generation}{Area\ (N+1)th\ generation} = \frac{1}{[\cos(B)]^2} = \frac{1}{[\cos(\pi/n)]^2}$$

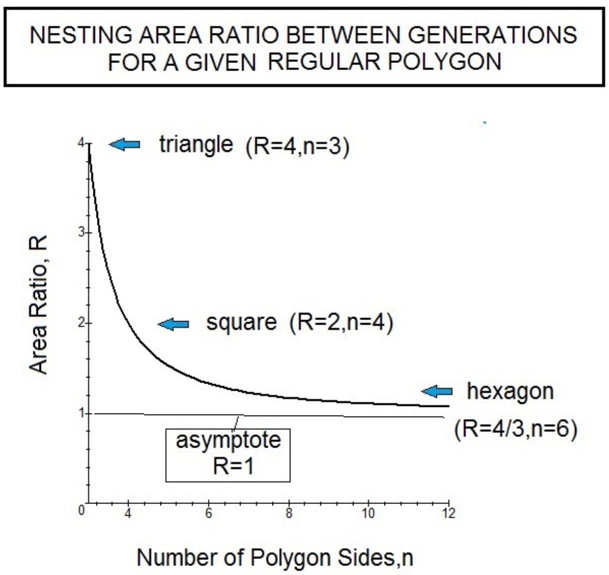
For the triangle, square, and hexagon one recovers the earlier results 4, 2, and 4/3, respectively. For a pentagon (n=5) we have –

$$R = \frac{1}{[\cos(\pi/5)]^2} = 2(3-\sqrt{5}) = 1.527864...$$

and for an octagon(n=8) we find-

$$R = \frac{1}{[\cos(\pi/8)]^2} = 2(2-\sqrt{2}) = 1.171572...$$

Also as n gets very large the area ratio R goes to one. A plot of R versus n follows-



Note that in most cases the extension of nesting to 3D polyhedra , where vertices touch the center of each of the previous generation faces, does not work. This is because the number of vertices of generation (N+1) will exceed the number of faces of the previous generation N. Thus, for example, the cube has eight vertices but only six faces. The only exception to this observation occurs for a regular tetrahedron where the vertices and the faces both equal four.

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