VALUES OF THE NUMBER FRACTION $f(N)$ WHEN $N>>1$

In several recent articles we have discussed the properties of a new function-

$$f(N) = \frac{\sigma(N) - (N + 1)}{N}$$

, where $\sigma(N)$ is the divisor function representing the sum of all divisors of the integer $N$. One can write the value of $\sigma(N)$ as-

$$\sigma(N) = \prod_{n=1}^{m} \left( \frac{p_n^{a_n+1} - 1}{p_n - 1} \right)$$

, where $p_n$s are the prime factors of $N$, $a_n$ the exponent of $p_n$, and $m$ the total number of prime factors in $N$. We term the function $f(N)$ the **Number Fraction**. It has the interesting property that it vanishes whenever $N$ is a prime number but is always positive when $N$ is a composite. When $N=p^n$, where $p$ is a prime and $n$ any positive integer, the number fraction reduces to-

$$f(p^n) = \frac{(1 - p^{1-n})}{(p - 1)}$$

This number approaches $1/p$ for $n \geq 2$ as $p$ gets large. For the special case of $N=2^n$ we have-

$$f(2^n) = 1 - \frac{2}{2^n}$$

, which approaches a value of one as $n$ gets large.

A question which now arises is –‘Does the value of $f(N)$ have a maximum or not as $N$ gets large?’ . The above two examples suggest it may have, however, one can not be sure. The simplest way to test things out is to actually evaluate $f(N)$ over a range of $N$s to see how large $f(N)$ becomes in that range. Marking down some of the local maxima of $f(N)$ found over the range $3<N<500,000$, we have the results-

<table>
<thead>
<tr>
<th>$N$</th>
<th>ifactor(N)</th>
<th>$f(N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>$2^2 \cdot 3 \cdot 5$</td>
<td>1.7833</td>
</tr>
<tr>
<td>180</td>
<td>$2^2 \cdot 3^2 \cdot 5$</td>
<td>2.0277</td>
</tr>
<tr>
<td>840</td>
<td>$2^4 \cdot 3 \cdot 5 \cdot 7$</td>
<td>2.4273</td>
</tr>
<tr>
<td>2520</td>
<td>$2^3 \cdot 3^2 \cdot 5 \cdot 7$</td>
<td>2.7138</td>
</tr>
<tr>
<td>15120</td>
<td>$2^4 \cdot 3^3 \cdot 5 \cdot 7$</td>
<td>2.9364</td>
</tr>
<tr>
<td>443520</td>
<td>$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$</td>
<td>3.3051</td>
</tr>
</tbody>
</table>
This table shows that the local maxima of \( f(N) \) are slowly increasing functions of \( N \). Furthermore the value of \( f(N) \) will have local maxima when the ifactor of \( N \) contains all the lowest prime numbers through \( m \) with decreasing powers \( p_n \). We may define a new number \( N(m) \) whose value approximates the prime product forms shown in the table. This number reads-

\[
N(m) = \prod_{n=1}^{m} (p_n)^{m+1-n} = 2^m \cdot 3^{m-1} \cdot 5^{m-2} \ldots
\]

Thus –

\[
N(8) = 2^8 \cdot 3^7 \cdot 5^6 \cdot 7^5 \cdot 11^4 \cdot 13^3 \cdot 17^2 \cdot 19^1 = 25968760179275365452000000
\]

which yields the value \( f(8) = 4.817111172\ldots \). Going to an even larger sixty-eight digit number \( N(12) \), we find \( f(N)= 5.718018580\ldots \). It is not clear from these last two results that the local \( f(N) \) maxima will become unbounded as \( N \to \infty \).

As already pointed out in an earlier note, the local maxima in \( f(N) \) are represented by numbers divisible by \( 6=2 \cdot 3 \). This fact is confirmed by the above table. Also one finds that \( N+1 \) or \( N-1 \) are often prime numbers. In the above case for \( N(12) \) we find \([N(12)+1]\) is a prime. It reads-

\[
p = 55784440720968513813368002533861454979548176771615744085560000000001
\]

Other large primes are found for \( N(15)+1 \), \( N(33)-1 \), and \( N(35)+1 \).

To demonstrate the local maximum property of \( f(N) \), we present a graph of the number fraction in the neighborhood of \( N(5) \). Here is the figure-
Note the sharp peak in the value of $f(N)$ at $N(5)=174636000$ and the prime at $N=N(5)+1$. The local maximum in graphs like this one show that $N$ is there equal to a super-composite having a large number of factors. The factors for $N(3)=2^3 \cdot 3^2 \cdot 5^1 = 360$ are:

\{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360\}

producing $f(360)=2.24722\ldots$

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