EXPRESSING NUMBERS USING VARIOUS BASES

When expressing any integer N it is mainly done using a base ten numbering system. This system involves ten characters 0-1-2-3-4-5-6-7-8-9 and was undoubtedly developed historically because early man used his ten fingers to count. There are of course also other number systems including the important binary system using a base two and requiring only the symbols 0 and 1. We want here to show how one can express numbers in various bases and then show how one carries out the basic operations of addition, subtraction, multiplication, and division.

We begin with the decimal system which uses ten as its base. In this system a number-

$$N = 3108 = +3x10^3 + 1x10^2 + 0x10^1 + 8x10^0$$

That is, 3 thousands+1 hundred+0 tens + 8 ones make up the number. So for abbreviation sake one simply writes down in order the numbers which multiply the various powers of ten. To add two numbers in decimal we simply add up the numbers multiplying various powers of ten. That is-

$$37+81=(3+8)x10^1+(7+1)x10^0=11tens+8ones=1hundred+1ten+8ones=118$$

To subtract we have-

$$87-49=4 \text{ tens}-2 \text{ ones}=3 \text{ tens}+8 \text{ ones}=38$$

Multiplication and division follow by operating with the various powers of ten. So-

$$23x46 = (2x4)x10^2 + (12+12)x10^1 + 18x10^0 = 10^3 + 0x10^2 + 5x10^1 + 8x10^0 = 1058$$

and-

1

$$135/9=\{1x10^2+3x10^2+5x10^1\}/\{9x10^0\}=1x10^1+5x1^0=45$$

The next number system is the binary one. It was first developed by G.W. Leibnitz some three hundred years ago and has turned out to be essentially the system used in all electronic computer calculations. The binary system is based upon the powers of two and uses only 1 and 0 as symbols. To write down a number N in binary we expand a typical number given in decimal such as-

$$37 \rightarrow 1x2^5+0x2^4+0x2^3+1x2^2+0x2^1+1x2^0=10010$$

The binary version of 37 is thus read off by writing down the symbol multiplying the power of twos in descending order. It is very easy to generate these numbers by hand and even easier to make use of our MAPLE commandconvert(N,binary);

Here N is given in decimal and we are converting to binary. One can go between any two bases by use of the command-

convert(N_{b1},b_2,b_1);

Thus we have, for example, that-

convert(10100110011,decimal,binary); =1331

Here we have carried out the reverse procedure of going from a binary number to its equivalent decimal form. A quick table using the conversion operation –

convert(N, binary, decimal);

follows-

Decimal N	**Binary N**
0	0
1	1
2	10
3	11
4	100
5	101
6	110
7	111
8	1000
9	1001
10	1010

From this table we see at once that two to a given power n has the equivalent binary form 1 followed by n zeros, That is, 64→1000000 and 1024→10000000000 . Also one notes that doubling a number just adds a zero at the end of the number. The addition of two numbers in binary, say 3 and 6 reads 11+110= 1001. That is-

0+0=0 , 0+1=1 and 1+1=10

so that
```
 111
 +11
---------
1 0 1 0
```

The subtraction rule is seen to be- 0-0=0 , 1-0=1 , and 1-1=0

So that 1001
- 101

$$\begin{array}{r} 1001 \\ -\ \ 101 \\ \hline 1\,0\,0 \end{array}$$

Thus any number in decimal can be represented in binary. To treat decimal points in binary we look at the negative powers of two represented in binary. Thus 1/2→0.1, 1/4→0.01, and 1/8→0.001 . So $1/2^n$→0.00….001, where the number of zeros after the decimal point equals n-1. Also the decimal number such as 2.456 converts to 10.01110100 in binary.

In looking at both the decimal and binary form of representing numbers one can see that a large base b requires more characters to represent a number but offers the advantage that the length of a given number will be shorter than a base with fewer characters . We have constructed a small table which gives the length of a number in decimals compared to that in binary. Here is the result-

Numberof digits in Binary	Number of digits in Decimal	Ratio of digits decimal to digits binary
3	1	3
6	2	3
9	3	3
13	4	3.25
16	5	3.20
19	6	3.1666…
23	7	3.2857…

Although not exact, the table indicates that digit length of a binary number is about three times longer than its decimal counterpart and that this ratio increases very slowly as the decimal length becomes even larger. The number 2^{300} has 91 digits in decimal and 301 digits in binary. The ratio between the two is 3.3076.. and so lies slightly below log(10)/log(2)=3.3219 which appears to be the upper limit.

Another often used number base is the hexadecimal system. It expresses numbers in powers of 16 and usually uses the following sixteen symbols-

{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A ,B, C, D, E, F }

to represent a number. Consider the following identity-

$357=1x16^2+6x16^1 +5x\ 16^0$ Its hexadecimal form therefore reads 165

One of the advantages offered by the hexadecimal system is that $16=2^4$ and so can easily relate to the binary system and hence be of use in computer operations by reducing storage requirements. For example the hexadecimal notation 165 with 3 symbols can be used to represent 101100101 in binary form. This works as follows-

$165 \rightarrow 16^2+6x16^1+5x16^0=2^8+2^6+2^5+2^2+2^0=101100101$

The digit length ratio between a hexadecimal base and a decimal base is about log(10)/log(16)=0.8304. The ratio between a binary to hexadecimal base digit length is log16/log2=4. Some even larger bases relying on powers of two are the bases 32 and 64. Their use in computer technology is clear. Again the advantage of such power of two bases is that it allows storage of large amounts of binary data in a much more compact form.

There are many additional number base systems but these play only historical roles and are no longer in common use. These include the base 20 system (vigesimal) of the Maya undoubtedly established by counting of ten fingers and ten toes. Also the ancient Sumarians and later the Babylonians used a base 60 system (sexagesimal) probably based upon the solar year of 365+1/4 days. The number 60 can be considered as a rich number since the number of divisors 1,2,3,4,5,6,10,12,15,20,30,60 is large.This makes the number 60 ideal for expressing numbers which are fractions and multiples of sixty. The sexagesimal system is partially retained to the present day in angle measurements and in the relation between hours, minutes and seconds. Both base twenty and sixty systems are now obsolete for mathematical manipulations required in everyday commerce. Already the base ten system of the ancient Romans using only combinations of the symbols I,V,X,L for the first ninety-nine integers was quite impractical for carrying out multiplications and divisions. Their system also lacked the concept of zero which was not introduced into Europe from the Middle East until about 12 hundred AD.

Summarizing we can say that any number N can be expressed in a base b system as -

$$N = \sum_{n=p}^{0} a_n b^n = a_p b^p + a_{p-1} b^{p-1} + ... + a_0 b^0$$

So we have, for example, that-

$$2x10^1+4x10^0=1x2^4+1x2^3=4x6^1+0x6^0$$

In abbreviated form this number then reads 24 in decimal, 11000 in binary, and 40 in base six(heximal). A base six system has so far found only very limited application. This may change in view of our earlier work on integer spirals where all prime numbers p of five and above satisfy the condition that p mod(6)=±1. All Mersenne Primes $M[p]=2^p-1$, with p >3, have the form 6n+1 and all Fermat Primes F[N]=2^2^N+1, with N≥1, have the form 6n-1.

U.H.Kurzweg
Dec. 19, 2016