EXPRESSING AND MANIPULATING NUMBERS IN DIFFERENT BASES

Any real number with no terms to the right of a decimal point can always be written as-

$$N=\sum_{k=0}^{n-1}a_k b^k=a_0+a_1 b+a_2 b^2+....+a_{n-1}b^{n-1}$$

, where b is the base being used and a_k are the elements a_0 through a_{n-1}. The most common number representation involves the base b=10 where we have the ten elements a_k={0,1,2,3,4,5,6,7,8,9}. So, for example, we can write-

$$3487=3x10^3+4x10^2+8x10^1+7x10^0$$

So this number , expressed in the *decimal system* , will be just the concatenation of the a_ks={3-4-8-7}. Such a base ten system has its historical origin most likely due to finger counting by early man. It is the system individuals first learn about in pre-school and elementary public schools and is the system which most individuals deal with exclusively in their everyday lives. There is of course no reason that the decimal system should be the only number system of importance. For example, an alien race with just three fingers on each of its two hands would probably come up with a base b=6 number system. The ancient Mayans used a base twenty system (b=20) obviously originating with finger and toe counting. Electronic computers use a *binary system* (b=2) exclusively because of the on-off nature of their processing procedures. In binary we have just two 'a's of 0 or 1. One can write the decimal base number N=37 as-

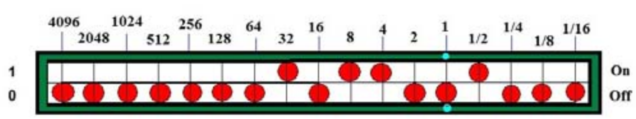
N= $37=2^5+2^2+2^0$

So that reading off the 'a's we get the binary representation –

N=100101

Several years ago I designed a binary abacus which allows one to add and subtract binary numbers. Here is a schematic of the device showing the decimal number N=44.5 in its binary form 101100.1 -

BINARY ABACUS SHOWING THE NUMBER 44 1/2=101100.1



Doubling a number just adds a 0 to the right of its binary sequence. Thus 23=10111 with 46=101110 and 94=1011100

Note here that we have also included a term to the right of the decimal point. To become proficient with operating this type of binary abacus it pays to commit to memory the first ten powers of 2.

The other day I ask several of my colleagues during our weekly luncheon meeting what is the next number in the sequence-

2-4-16-65536

At first they were stumped, but then most recognized that one is dealing with –

$$2^1 - 2^2 - 2^4 - 2^{16}$$

So that the next number must have the huge value of 2 taken to the power 2^{16} . In binary form this sequence reads-

10 – 100 – 10000 -10000000000000000-

Most advanced math programs have a built in computer command which takes one from decimal to any other base. In particular to go from decimal to binary one applies the command-

convert(N, binary, decimal);

Using this command, we find the following conversions to binary for the decimal numbers N= 1 through 40 -

Decimal(b=10)	Binary(b=2)	Decimal(b=10)	Binary(b=2)
1	1	21	10101
2	10	22	10110
3	11	23	10111
4	100	24	11000
5	101	25	11001
6	110	26	11010
7	111	27	11011
8	1000	28	11100
9	1001	29	11101
10	1010	30	11110
11	1011	31	11111
12	1100	32	100000
13	1101	33	100001
14	1110	34	100010
15	1111	35	100011
16	10000	36	100100
17	10001	37	100101
18	10010	38	100110

19	10011	39	100111
20	10100	40	101000

You will notice certain properties of the binary forms. First of all they have about three times as many digits as the corresponding decimal form. Secondly, adding a zero to the right doubles the number. Thus 17 has 10001 while 34 yields 100010. If N equals the pth power of 2 then its binary form reads 1 followed by p zeros. Thus $32=2^5$ reads 100000 in binary. Even numbers end in 0 while odd numbers end in 1. A number $N= 2^p-1$ has a binary form consisting of p ones with no zeroes. Thus, for example, the Mersenne prime $2^{13}-1=8191$ reads 1111111111111 in binary.

The addition rules in binary are 1+1=10 and subtraction is governed by 10-1=1. So N=5+7 reads 101+111= 1100 and N=9-3 reads 1001-11=110. Going back to the binary abacus shown above one sees at once that 44+24 is given by moving the ½ ,8 and 32 bead down and moving the 64 bead up. The result now reads 1000100 or 68 in decimal.

Multiplication follows from 1x1=1 and 10x1=10. Thus N=12x3 reads 1100x11=11000+1100=100100.

In addition to binary, a base in common use in connection with electronic computing is the hexadecimal system whose base is b=16. Its sixteen elements are-

a={0 1 2 3 4 5 6 7 8 9 A B C D E F}

It has six more elements than decimal and so will have shorter digit forms for a number. For instance, the decimal number N=47218 reads B872 in hexadecimal and N=189335 reads 2E397. Note that $N=16^p$ equals 1 followed by p zeros in hexadecimal. Thus $N=65536=16^4$ is written as 10000.

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