SOME NUMBER SEQUENCES CREATED BY THE INTEGERS

If one looks at all integers in an ascending order expressed in a decimal base one has the sequence-

\[ S = \{1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \ 16 \ 17 \ 18 \ 19 \ 20 \ ... \} \]

There are multiple ways to moderate this sequence. For example, one can write-

\[ F = \{1 \ 2 \ 3 \ 5 \ 8 \ 13 \ 21 \ 34 \ 55 \ 89 \ 144 \ 233 \ ... \} \]

Here the nth tern equals the sum of the n-1 and n-2 term. Thus 21+34=55. This is the famous Fibonacci Sequence. The generating formula for its elements is given by-

\[ f(n+2) = f(n+1) + f(n) \text{ with } f(1) = 1 \text{ and } f(2) = 2 \]

The element ratio \( f(n+1)/f(n) \) goes to the Golden Ratio \( \varphi = \frac{1+\sqrt{5}}{2} = 1.61803398... \) as \( n \) goes to infinity.

Another variation of the sequence \( S \) is-

\[ T = \{1 \ 2 \ -1 \ 3 \ -4 \ 7 \ 11 \ -4 \ 7 \ 11 \ -4 \ 15 \ -19 \ ... \} \]

Here an inspection of the elements yields the generating formula-

\[ f(n+2) = f(n) - f(n+1) \text{ with } f(1) = 1 \text{ and } f(2) = 2 \]

which can be taken to any desired number of elements by use of the one line computer program

\[
\text{for } n \text{ from 1 to 10 do } f[n+2]:=f[n]-f[n+1] \text{ od;}
\]

One finds the ratio \( f[100]/f[99] \approx -1.61803398874989482045868343656381177203.. \)

indicating that the ratio as \( n \) goes to infinity is just the negative of the Golden Ratio.

There are an infinite number of other generating formulas which transform the original integer sequence into other forms. We could take-

\[ f(n+3) = f(n+2) + f(n+1) + f(n) \text{ subject to } f(1) = 1, \ f(2) = 2 \text{ and } f(3) = 3 \]

This produces the sequence –

\[ U = \{1 \ 2 \ 3 \ 6 \ 11 \ 20 \ 37 \ 68 \ 125 \ 230 \ ... \} \]
Here the ratio \( f[(n+1)/f[n] \) as \( n \) goes to infinity equals \( 1.839286755214 \). This last sequence can be thought of as a modified Fibonacci Sequence where the ratio of the \((n+1)\) element to the \(n\)th element as \( n \to \infty \) equals a new irrational constant. See if you can figure out what combination of roots of the integers produce this constant.

Other sequences follow directly from the generating formula-

\[
f[n+1]=F\{ n,f[n]\} \text{ subject specified initial conditions.}
\]

A sequence following from this general form is-

\[
f[n+1]=(f[n])^2+n \quad \text{with } f[1]=1
\]

Its explicit form reads-

\[
V=\{1 2 6 39 1525 2325630 \ldots\}
\]

Here elements approach infinity very rapidly after starting out slowly in size.

Consider next the generating formula-

\[
f[n+1]=f[n]+n+1 \text{ subject to } f[1]=1
\]

Here -

\[
\begin{align*}
f[2] &= 3 = \frac{2 \cdot 3}{2} \\
f[3] &= 6 = \frac{3 \cdot 4}{2} \\
f[4] &= 10 = \frac{4 \cdot 5}{2}
\end{align*}
\]

From this we can surmise that-

\[
f[n] = \frac{n \cdot (n + 1)}{2} = \sum_{k=1}^{n} k
\]

so that the \(n\)th element in the sequence will be the sum of the first \(n\) integers. We can also treat the above generating function as a finite difference equation whose solution is \( f[n]=n(n+1)/2 \). The sequence here reads-

\[
W=\{1 3 6 10 15 21 28 36 46 \ldots\}
\]
It is not always necessary that the elements based on the original number sequence S have integer elements. For example, the generating function:

\[ f[n+1] = 1 + 1/(1+f[n]) \] subject to \( f[1] = 1 \)

tyields the sequence:

\[ R = \{1, 3/2, 7/5, 17/12, 41/29, 99/70, \ldots\} \]

Here the elements approach the limiting value of \( \sqrt{2} = 1.414213562\ldots \) as \( n \) gets large.

Finally, let us consider a sequence which has mostly complex elements. We look at:

\[ f[n+1] = i^{f[n]} = \exp(i\pi f[n]/2) \] subject to \( f[1] = 1 \)

Here the sequence generated reads:

\[ M = \{1, i, \exp(-\pi/2), (0.9471589980+0.320764450i), \ldots\} \]

A plot of all the elements through \( n=34 \) produces the following interesting three prong spiral pattern in the complex \( z=x+iy \) plane-

As \( n \) goes to infinity, the three spirals are seen to merge to a value of \( z=x+iy \) given by solving:

\[ z = i^z \]

The solution is expressible as a Lambert function and reads-
\[ z = \frac{2i}{\pi} LambertW(-\frac{i\pi}{2}) = 0.43828 + i0.36059 \]

U.H.Kurzweg  
March 5, 2019  
Gainesville, Florida